Paraconsistent absolute generality.

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Declaration pursuant to the *Regulations for Philosophy in all Honour Schools including Philosophy*.

I declare that this thesis has the same title as that previously approved by the Faculty Board, that it is my own work, and that it has not already been submitted (wholly or substantially) for an Honour School other than one involving Philosophy, or another degree of this University, or a degree of any other institution.

Note.

This file is identical to that submitted to the examiners, but for the inclusion of this note, the removal of the candidate number and word-count from the cover, the inclusion of the date, and my name and college therein, and the following

Acknowledgements.

Alex Roberts (now across the pond) had the largest single influence on my undergraduate education; in his tutorials it first occurred to me that serious logic could not only be of interest but that I might have been sufficiently able to digest it. This was effected under the benign hand of Oliver Pooley, who, in my opinion, showed very good taste in tutors.

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Contents

Abstract.

Quantification over absolutely everything may be incoherent or impossible because it would lead to contradiction [‣W03: § IV; ‣S19: § 1.4]. Dialetheists seek to render contradictions, such as those arising from liar sentences, harmless [‣P06].

I offer a dialetheist defence of absolutely general quantification. It is motivated by paradoxes that arise from attempts to account for logical consequence [‣W03: § IV]. It differs from Priest's dialetheist defence of absolutely general quantification [‣P07] in that it does not endorse naïve set theory, and so can more naturally recapture classical reasoning in mathematics and set theory to which dialetheists need not and often do not object. It also therefore avoids problems in developing paraconsistent naïve set theory. Finally, it avoids expressibility problems that other defences of absolutism incur.

Notational conventions.

Clickable links are marked ‣thus. I write $\phi!$ for $\phi \wedge \neg\phi$; ! binds more closely than other connectives, including unary, i.e. $\neg\phi! =_{df} \neg(\phi \wedge \neg\phi)$.

1 Preliminaries.

- ▸§ 1 I expound generality absolutism and relativism.
- ▸§ 2 I outline arguments against absolutism, and explain my focus on objections from 'indefinite extensibility'.
- ▸§ 3 I introduce dialetheist defences of absolutism from such objections.
- ▸§ 4 I give a fuller plan of the thesis.

1 Generality absolutism and relativism.

1.1 *Absolutism.* To utter such quantifiers as 'everything', « toutes choses », and so on in natural language is so commonplace as to be unremarkable. In many contexts, quantifiers are restricted. If I say 'everything is in the suitcase', I do not mean that *I* am in the suitcase, or the Radcliffe Camera. In that context, 'everything' was implicitly restricted [▸W03: 415]—perhaps to the relevant items of luggage. Clearly not '*everything*—everything in the entire universe' is in the suitcase: the quantifier has a 'tacit restriction to a domain of contextually relevant objects' [▸W03: 415].

Absolutely general quantification is not thus restricted. Absolutely general quantifiers range over 'absolutely everything whatsoever in the entire universe. [*Thumping the table:*] NO EXCEPTIONS!' [▸S19: 1]. So a theologian might say that god understands everything; and a philosopher might say that 'everything is mereologically simple' [▸S19: 3]. The theologian does not mean that god understands everything in the suitcase, or in the library, or on earth; the philosopher does not mean that merely quarks or gluons are mereologically simple; for 'NO EXCEPTIONS' are admissible.

Absolutism, to Studd, is to a first approximation the view that 'sometimes… quantifiers such as "everything" or $\forall x$ range over an absolutely comprehensive domain'—one that is unrestricted in the way just described [▸S19: 1]. I would impose a further requirement. Opponents of absolutism often charge that absolutely general quantification is impossible *on pain of triviality*. That is, in a system of reasoning in which absolutely general quantification is possible, we may seem to be able to make sense of the absolute generality of the candidate quantifier—but, often making use of that generality, there will be a sound argument for any conclusion. This is a price all absolutists, to my knowledge, would find intolerable, and rightly so. The absolutist must therefore refute the charge of triviality.

1.2 *Generality relativism* negates absolutism. The debate to which this thesis is ultimately addressed concerns the question:

Q1 *Should we be absolutists or relativists?*

Tellingly, relativists expend much effort developing acceptable positive articulations of their view. Here is one illustration of the sorts of difficulties to be expected [▸w03: § v].

The relativist claims that

(1) it is impossible to quantify over everything.

Therefore, they must admit that

(2) I am not quantifying over everything,

and so,

(3) something is not being quantified over by me.

▸(3) is said at some time t_0.[1] If speaking truly, by standard principles of semantic ascent,

(4) 'Something is not being quantified over by me' is true as uttered by the relativist at t_0.

In construing ▸(4), it is plausible that

(5) 'something Fs' is true as uttered by *s* at *t* iff something over which *s* quantifies at *t* satisfies '*Fs*' as uttered by *s* at *t*; and

(6) something satisfies 'is not being quantified over by me' as uttered by *s* at *t* iff it is not quantified over by *s* at *t*.

From ▸(4) and ▸(5),

(7) something over which the relativist quantifies at t_0 satisfies 'is not being quantified over by me' as uttered by the relativist at t_0.

From ▸(6) and ▸(7),

(8) something over which the relativist quantifies at t_0 is not quantified over by the relativist at t_0.

1. We could add various other features of the linguistic context; the effect of the example would remain the same.

Hence perhaps the most obvious way of positively articulating relativism leads to contradiction. The problem is sufficiently widespread that one absolutist argument is that relativism is afflicted by a fatal ineffability.[2] However, I shall not explore that question. This thesis is intended to indirectly contribute to the debate on ‣Q1: it explores neglected (dialetheist) forms of absolutism, and whether they have any attraction by absolutist lights. Even if absolutism proves to be strongest in dialetheist form, it will remain to compare it with relativism. For relativists may well be able to overcome objections from ineffability, and, indeed, to articulate an attractive positive position. The ultimate implications of the arguments made in this thesis for the wider absolutist–relativist debate rest on such considerations, which I leave beyond the scope of this thesis.

2 Objections to absolutism.

Absolutism is often taken to be *prima facie* attractive [‣S19: vii, 1–3; ‣F14: 442; ‣W03: §§ 1–3]. Absolutely general quantification appears to arise in ontology, metametaphysics, philosophy of logic, philosophy of mathematics [‣R20: § 1], and even everyday theorising [‣W03: § V], so retaining it, if possible, would be desirable. The question arises:

Q2 *What motivates relativism?*

In view of the *prima facie* attractions of absolutism, the answer to ‣Q2 is generally negative: objections to absolutism motivate relativism.

I agree with Studd that 'considerations from indefinite extensibility provide by far and away the most powerful case against absolutism' [‣S19: 4].[3] This is a widespread but perhaps not undisputed view [see e.g. ‣R20]. I shall assume it for two reasons. The first is simply to limit the scope of this thesis. The second is that dialetheism does not appear likely to help, as I shall argue in ‣¶ 2.1.

Therefore, in this thesis, I shall concern myself primarily with objections from indefinite extensibility.[4] If that is right, ‣Q1 is settled by the following:

Q3 *Do objections from indefinite extensibility justify relativism?*

2. The matter has been ventilated extensively [see *i.a.* ‣F⁺21: § 11.4; ‣S19: §§ 1.6 and 5.1; ‣W03: § V; ‣F14: § 2].
3. Florio calls these '[t]he most powerful arguments' [‣F14: § 3.1].
4. Rayo and Uzquiano, in their introduction to the first compendium on absolute generality [‣R⁺07: §§ 1.2.1–2], distinguish objections from the 'all-in-one' principle, but they are closely related, and I shall not insist on the distinction.

I introduce these objections in ‣¶ 2.2. In ‣§ 3 I explain how dialetheism can help.

2.1 *Objections disregarded.* I more fully characterise dialetheism in ‣§ 3, but the central element of any dialetheist defence of absolutism (or any other position) is that contradiction, *per se*, need not be harmful. Objections from *contradiction* are amenable to dialetheist responses; direct counterarguments are not. In order to justify my emphasis on objections from indefinite extensibility, I shall now explain how two other sorts of objections are more in the vein of direct counterarguments than *reductiones ad contradictione*.

2.1.1 *Objections from sortal restriction* [‣s19: 4–6]. In many languages, quantifiers combine a determiner ('every', « quelque », and so on) with a nominal (e.g., 'thing', « un », « libro »): thus 'everything', « quelqu'un ». A proposed absolutely general quantifier must then have a nominal that 'applies indiscriminately to any item whatsoever, regardless of its sort'. The objection runs as follows. First, quantifiers' nominals must be sortal. Second, we should be able to count sortals: to answer such questions as how many books there are in a room. Third, to count such items, there must be a 'non-trivial criterion of identity…for items' of each sortal. Fourth, putatively universal sortals have no such criterion. Therefore, universal quantification is impossible.

It is not clear what purpose exactly a dialetheist response to this objection would serve. Dialetheists typically seek to revive positions that are targeted because they putatively lead to contradiction—to them, contradiction *per se* is not a problem. But if either side of the contradiction is objectionable *per se*, merely showing that some contradictions are harmless is insufficient. Moreover there is no obvious contradiction to explain away as harmless. There is little attraction in the view, for example, that putatively universal sortals both do and do not have a non-trivial criterion of identity.

2.1.2 *Objections from metaphysical relativism* [‣s19: § 1.3]. Suppose that there are two isolated linguistic communities. One comprises mereological nihilists; another mereological universalists. The nihilists deny that there are ordinary objects such as chairs; all that exists is mereologically simple. The universalists not only accept the existence of ordinary objects but also posit that there is a mereological fusion of any one or more things.

Then suppose that they meet. The nihilists will claim that 'nothing is non-simple'; the universalists will reply that 'something is non-simple'. Is there a genuine dispute?

A metaphysical relativist would say that there is not. Members of each linguistic community 'speak truly relative to their linguistic framework/conceptual scheme/language in virtue of operating with different interpretations of the existential quantifier'. The relativist also denies that any particular interpretation is privileged, in the sense that it singles out what 'really exists'—the interpretation 'God would use', that 'carves nature at the joints' and so on.

The charge against absolute generality is that it cannot accommodate this sort of relativism, because an absolutely general quantifier must be privileged. Thus,

> BIGGEST IS BEST. If there is an absolutely general existential-quantifier-interpretation, it is the unique metaphysically privileged/maximally joint-carving existential quantifier-interpretation.

Just as the objection from sortal restriction does not rely on the objectionability of contradictions, so too the putative argument from metaphysical relativism here is not simply that generality absolutism leads to some undesirable contradiction; it is that it leads to an undesirable *consequence*, namely metaphysical anti-relativism. There is no obvious contradiction to accept in sight.

2.1.3 *Recapitulation.* I disregard objections from sortal restriction and metaphysical relativism for two reasons. First, they do not appear to be decisive. Second, and more importantly, there is no obvious dialetheist response to them.

2.2 *Objections from indefinite extensibility* are the concern of most of the thesis.

2.2.1 *The very idea.* Studd writes that [▸s19: 10]

> some concepts *F* are *indefinitely extensible*: to a first approximation, this is to say that given any domain comprising *F*s, however extensive, a further *F* can always be specified, giving rise to a wider domain.

It may be objected that such a definition of indefinite extensibility already supposes too much about domains. For example, does appeal to a 'wider' domain presuppose that domains are sets—what are domains, if not sets? If so, we might worry that this objection is unfair to the absolutist, who could articulate their position without reference to sets; it might too closely associate absolute generality with Russell's paradox. I shall address this issue in ‣§ 8, and set that question to one side.

Studd offers two *reductiones*: the Russell *reductio* purports to show that collections are indefinitely extensible, and the Williamson–Russell *reductio* purports to show that interpretations are indefinitely extensible.

2.2.2 Preliminaries to the Russell reductio [‣s19: 11]. We employ the term 'collections' roughly in accordance with its pre-theoretic use: they are 'arbitrary' and extensional, and comprise one or more members. In saying that collections are 'arbitrary', Studd means that 'there need be no non-arbitrary relation between the members [a collection] comprises'—they needn't 'be the property of a single collector... relevantly similar or metaphysically joint-carving... [or] specified by a formula of a formal language or a predicate of a natural one' [‣s19: 11]. We use $\in$ to indicate membership of a collection.

2.2.3 The Russell reductio [‣s19: 11]. Suppose we have quantifiers $\forall_D$ and $\exists_D$ ranging over some domain D. Then there is a collection r_D of all and only x such that $\exists_D y(y = x)$ (D ranges over x) such that $x \notin x$. We shall show that $\nexists_D y(y = r_D)$ (D does not range over r_D) by contradiction.

Assume that r_D either is or is not self-membered. If it is self-membered, then $r_D \in r_D$. By assumption (for contradiction), $\exists_D y(y = r_D)$. Hence by definition of r_D, $r_D \notin r_D$.

If it is not self-membered, then $r_D \notin r_D$. By the same assumption and the same definition, $r_D \in r_D$. Hence $r_D \in r_D$ just in case $r_D \notin r_D$. Therefore, D did not range over r_D, so there is a wider domain.

As a corollary, if D is absolutely general, $\exists_D y(y = r_D)$. But that leads to contradiction. So D is not absolutely general.

2.2.4 Preliminaries to the Williamson–Russell reductio [‣s19: 13]. Consider a first-order language with a unary predicate P. We should expect an interpretation of this language to specify to which things P applies: so we shall write $i \vDash Px$ when i interprets P to apply to x. We shall argue that interpretations are indefinitely extensible.

2.2.5 The Williamson–Russell reductio. Suppose we have quantifiers $\forall_D$ and $\exists_D$ ranging over some domain D. There is then an interpretation i_D on which $i_D \vDash Px$ just in case x is an interpretation in D such that $x \nvDash Px$.

Now, suppose that $\exists_D x(x = i_D)$ for contradiction. Then either $i_D \vDash Pi_D$ or $i_D \nvDash Pi_D$. If $i_D \vDash Pi_D$, by definition, $i_D \nvDash Pi_D$. If $i_D \nvDash Pi_D$, by definition, $i_D \vDash Pi_D$. Hence $i_D \vDash Pi_D$! Therefore, the initial supposition is false; D did not range over i_D. Therefore, a wider domain is available.

A similar corollary applies: if D is absolutely general, $\exists_D x(x = i_D)$. But that leads to contradiction. So D is not absolutely general.

2.2.6 *Conclusion.* Even if it is admitted that absolute generality leads to contradiction, the relativist must also show that it is an intolerable price for the absolutist to pay. The subject of this thesis is a *dialetheist* approach that seeks to render contradiction harmless.

3 Dialetheism and paraconsistency introduced.

3.1 *Dialetheia introduced.* First, an example. The proper treatment of liar sentences, such as 'this sentence is not true', is notoriously tricky; one view is that it is both true and not true. We can motivate such a view by the following informal reasoning: the sentence is true or not true; if it is true, it is not true; and if it is not true, it is not not true; therefore, it is not true and not not true—which might be taken to show that it is not true and true, if double negation elimination is admitted.

A *dialetheia* is a truth-bearer such that both it and its negation are true [**priest2023**].

I shall not concern myself with the difficult question of what, exactly, truth-bearers are. One candidate dialetheia, on the treatment above, is the liar sentence.

Dialetheism is the view that dialetheia exist.

3.2 *Paraconsistency introduced.* Why believe dialetheia don't exist? One reason is their facial absurdity. It is not obvious that facial absurdity should be decisive or given much weight. Even if it should, *de intuitionibus non est disputandum*—not much more can be said in that vein.

More worryingly, the following argument shows that dialetheists must give up something—classical logic—on pain of triviality.

Definition. The argument *ex contradictione quodlibet* (ECQ) is that $P, \neg P \vdash Q$ for arbitrary sentences P and Q.

Theorem. Classical logic validates *ex contradictione quodlibet*.

Proof.

$$\dfrac{\dfrac{P}{P \lor Q} \qquad \neg P}{Q}$$

Corollary. Under classical logic, if there are dialetheia, there is a sound argument for every sentence: simply substitute P and Q appropriately.

Since some sentences aren't true (e.g., $2 + 2 = 5$), we appear to have a *reductio ad absurdum*: we must reject dialetheism. But that is not the only solution. We must reject the combination of dialetheism and classical logic; but nothing we have seen so far makes the rejection of the latter untenable.

Definition. A logic is *explosive* if it validates *ex contradictione quodlibet*, and *paraconsistent* if it does not.

Classical logic is explosive, so classical logicians must avoid contradictions; dialetheists advocate paraconsistent logics and so can accept contradictions.

3.3 *Dialetheist absolutism.* I shall concern myself, in this thesis, with *dialetheist absolutism*, which, as the name suggests, combines dialetheism and absolutism. The dialetheia whose existence the dialetheist absolutist accepts are, in particular, those that the relativist can legitimately derive from the possibility of absolutely general quantification. (The dialetheist absolutist need not accept all attempted derivations of contradiction from the possibility of absolutely general quantification.)

The dialetheist absolutist therefore answers as follows.

▸Q1 We should be absolutists.

▸Q2 Objections from indefinite extensibility are the primary motive for absolutism.

▸Q3 But they do not succeed, because they erroneously assume that there are no true contradictions.

Dialetheists in general accept some derivations of contradictions; but, instead of taking them to be *absurda* that underpin *reductiones*, it instead takes the contradictions to be true. What are those contradictions in the debate over absolute generality?

3.3.1 *Russell* reductio. We derive from the absolute generality of a domain D that $r_D \in r_D$! Every step of the argument is accepted except the conclusion that D is not absolutely general. That is: the domain D can be taken to be absolutely general, r_D exists, and $r_D \in r_D$!.

3.3.2 *Williamson–Russell reductio.* Similarly, we accept that D is absolutely general, that i_D exists, and that $i_D \vDash Pi_D$!

3.4 *Paraconsistent logic, contradictions, and dialetheia.* Not *all* contradictions have dialetheia as their conjuncts. (Otherwise, we end up with triviality again.) So, which are? The classical logician has a very easy answer: none. The ease of that answer is admittedly not wholly desirable, since it is forced upon them on pain of triviality.

The dialetheist, on the other hand, has a trickier task, since the two easy answers (all and none) are ruled out; but dialetheists claim this has substantial benefits.

In the case of the liar paradox, the benefit sought is a truth predicate that behaves naïvely—roughly as it does in natural language. The truth predicate appears to be applicable to arbitrary sentences and to conform to convention T. Classical logicians must accept, for example, the inelegance of a metalinguistic hierarchy and a hierarchy of truth predicates associated with each language. But it is folklore that it suffers from expressive limitations on pain of inconsistency—for example, untruth in a 'fixed point' in the hierarchy seems to be describable formally and even set-theoretically, but is inexpressible on pain of triviality [▸B07].

In the case of absolute generality, similar considerations apply. Relativism allegedly incurs expressibility deficits (*inter alia*); absolutism in combination with various other *desiderata* leads to contradiction; therefore, to avoid expressibility deficits, we must accept a contradiction, and so, to avoid triviality, paraconsistent logic. The criterion by which we should assess whether to accept a contradiction is to what extent we are committed to the premisses by which a contradiction is reached; if we are strongly committed to them, acceptance of contradiction becomes more attractive.

I do not propose to determine what the exact price of absolutism is—practically every charge against the relativist has been contested in one place or another. As stated earlier, I seek to address the modester question of whether dialetheism has any attraction by absolutist lights. I shall therefore assume that absolutism *generally* is motivated; the alternatives to acceptance of contradiction I consider are all absolutist.

How wide are the implications of the dialetheist response? What is required is a profound revision to logic that excludes a number of typical argumentative strategies—most obviously, *reductio ad absurdum*. Moreover the dialetheist must do some work to explain which contradictions are true—for example, in deriving that $\sqrt{2}$ is irrational by contradiction, the dialetheist presumably does not want to simply accept the contradiction. Dialetheists propose to replace classical with a paraconsistent logic, motivated by paradox. The dialetheist response therefore appears to jeopardise vast swathes of acceptable classical reasoning that is not paraconsistently valid—such as arguments by contradiction in mathematics. Some non-classical logicians often accept some classical inferences that their own logics are too weak to support, including Priest [‣P06: 110].[5]

> Perhaps no one except the most hardened classicist would mourn the loss of paradoxes of implication such as *ex contradictione quodlibet*; but the loss goes beyond these. For classical principles of inference that do appear to be used quite commonly are dialetheically invalid. The most obvious of these is material detachment, or, as it is commonly called, the disjunctive syllogism $\{\alpha \wedge (\neg\alpha \vee \beta)\} \vdash \beta$.
>
> More generally, suppose we call an inference *quasi-valid* if it involves essentially only extensional connectives and quantifiers, and is classically valid but dialetheically invalid.

Priest proceeds to argue that we can *recapture* quasi-valid reasoning. Another approach is to begin from scratch in a paraconsistent logic, without particularly insisting on recapturing classical results. I shall discuss such issues in ‣§ 5 and ‣§ 9.

4 The shape of the debate; analytic table of contents.

4.1 *Varieties of (dialetheist) absolutism. Some questions introduced.* Absolutism, in ‣¶ 1.1, was characterised somewhat circularly: the view that some quantifiers are absolutely general or unrestricted—which is to say, that the range over absolutely everything, and that their availability does not lead to triviality. This may be found a little unsatisfactory. A rather pressing and potentially decisive question is therefore:

Q4 *What does it mean to quantify? And what does it mean to quantify* absolutely generally*?*

This is of course a question that absolutists and relativists alike must answer (or dodge), whether or not they are dialetheists. Those who take a dialetheist path must then answer some further questions.

Q5 *Which of the putative contradictions derived from absolutism are true? Why? Which contradictions, in general, are true?*

The answer individuates various different forms of dialetheist absolutism, with different implications, costs, and benefits. Those

5. For a wider survey, see [‣M⁺20: § 4].

costs and benefits may help in choosing an answer to ‣Q4. If it can successfully be explained *why* the relevant contradictions are true, the dialetheist absolutist has won a partial victory; but, as noted above, such an argument would not directly settle the absolutist–relativist debate. But it is not only in individuating forms of dialetheist absolutism that this question is important: as we saw, dialetheists must generally explain which contradictions they take to be true, and why.

There is a closely related question:

Q6 *What should be made of quasi-valid reasoning?*

Acceptance of some contradictions as true will restrict attitudes to quasi-valid reasoning. For example, if quantified statements amount to statements about some corresponding set, absolutely generally quantified statements will have to amount to statements about some universal set (‣Q4): existentially quantified statements will say that there is at least one element *of that set* of which something is the case. This commits us to acceptance of some set-theoretic antinomies (‣Q5) and therefore precludes wholesale acceptance of ZFC (‣Q6).

4.2 *A metasemantic fork in the road.* Warren makes a useful, if rough, distinction between two metasemantic theories (theories that in particular 'explain what it is that determines both sentential and subsentential semantic facts') [‣W17: 87].

> **Bottom Up:** Subsentential semantic facts are explanatorily prior to sentential semantic facts (ceteris paribus).
>
> **Top Down:** Sentential semantic facts are explanatorily prior to subsentential semantic facts (ceteris paribus).

I do not wish to take any particular view on Warren's broader claim that '[a]lthough mixed accounts are certainly possible, most metasemantic and metaconceptual theories fit neatly into one of these two categories' [‣W17: 87] or his enumeration of which theories fall under which approach. What I do wish to take from this discussion is that we can distinguish two forms of absolutism, corresponding to two approaches to ‣Q4, to be considered in dialetheist form in the next two chapters.

‣II *Subsentential absolutism* seeks to characterise absolutism in terms of semantic facts about absolutely general *quantifiers*. For example, it could associate a set with each quantifier. Semantic facts about quantified sentences would then follow, although I do not consider that task.

‣III *Sentential absolutism* seeks to characterise absolutism in terms of semantic facts about absolutely generally quantified *sentences* first. These include their truth conditions and their inferential relations. Subsentential semantic facts would then follow, although I do not consider that task.[6]

‣II *Subsentential dialetheist absolutism.* 'Bottom-up approaches to metasemantics need to provide an independent account of quantifier meanings' [‣W17: 88]. Perhaps that is not really the job of approaches to *meta*semantics but rather a responsibility of the *semantic* theories of those committed to bottom-up metasemantics; nevertheless, the point is well taken. I shall consider two such approaches.

‣§ 5 *Naïve set-theoretic dialetheist absolutism* holds that

‣Q4 quantified statements are statements about sets: for example, to universally quantify is to make a claim about every member of some set; for every quantifier, there is a set; and, in particular, there is a universal set corresponding to the absolutely general quantifier.

Naïve set-theoretic dialetheist absolutism seeks to secure a universal set through paraconsistent naïve set theory; accordingly

‣Q5 some set-theoretic contradictions, such as Russell's paradox, are true.

The viability of this position depends on the viability of paraconsistent naïve set theory. In common with much of the literature, I argue that paraconsistent naïve set theory appears to come at high cost; more attractive paraconsistent set theories meanwhile do not admit a universal set, and so are of little use on this approach.

I provisionally therefore answer:

‣Q4 to quantify is not just to make some claim about a set.

‣§ 6 *Plural dialetheist absolutism.* A second bottom-up approach is *plural absolutism*:

‣Q4 to quantify is to make a claim about a *plurality* (which needn't be coextensive with a set.)

Plural dialetheist absolutism is motivated by the view that

‣Q5 certain Cantorian paradoxes related to plurals are true (but we can remain neutral or reject e.g. Russell's paradox).

6. See e.g. [‣F⁺15].

I shall claim that dialetheist plural absolutism fares better than classical plural absolutism and dialetheist absolutism via a universal set, but is worryingly *ad hoc* in its response to ‣Q6.

‣III *Sentential dialetheist absolutism.* Bottom-up approaches appear to be able to avoid objections from indefinite extensibility only at high cost, and dialetheism doesn't help decisively (‣Q3). What about a top-down approach?

‣§ 7 *Sentential absolutism.* The first task is to articulate absolutism sententially. On the sentential approach, 'facts about a quantifier's domain are explained by facts about the semantic properties of whole sentences involving the quantifier, typically the truth conditions of these sentences' [‣W17: 88]. The question is then which semantic facts about absolutely generally quantified sentences explain their absolute generality. I reach roughly the following conclusion:

‣Q4 the meaning of quantified statements is to be understood by associating them with their truth conditions and inferential commitments.

‣§ 8 *The inescapability of contradiction.* It might be thought that sentential absolutism avoids objections from indefinite extensibility (‣Q3), and so there is no motive for dialetheism. I argue otherwise: sentential absolutists face indefinite extensibility if they seek to characterise logical consequence. Therefore,

‣Q5 truth theorists face the Williamson–Russell paradox, and inferentialists or proof-theoretic semantic theorists face a closely related paradox; we should accept the contradictions that follow.

Accordingly, dialetheism retains its attractions.

‣§ 9 *Set theory and recapture reconsidered.* What of quasi-valid reasoning (‣Q6)? I argue that the sentential absolutist is in no worse a position to derive mathematical and set-theoretic orthodoxy or to grant it special status than classical logicians, by using a paraconsistent and paracomplete form of ZFC. Strictly, this minimises which reasoning is quasi-valid, and simply rescues much of mathematical orthodoxy as straightforwardly valid (given suitable axioms). Therefore,

‣Q6 orthodox set theory and mathematics can be recaptured as *valid* rather than merely *quasi-valid*; recapturing reasoning of the latter kind is therefore relatively unimportant.

This argument is of general interest to all logical revisionists.

▸§ 10 *Conclusion.* I argue that sentential dialetheist absolutism is preferable to plural dialetheist absolutism, and I situate the contributions of this thesis in the context of the larger absolutist–relativist debate.

11 Subsentential dialetheist absolutism.

5 Naïve set-theoretic dialetheist absolutism.

5.1 *Existing practice.* Standard model-theoretic semantics regards quantifiers as ranging over set-domains. Thus, in a typical semantics textbooks for linguists [‣K11: 96] we read that

> a quantifier expresses a relation between sets. For example, *All ravens are black* expresses [that]... the set of ravens is completely included in the set of black things.

5.1.1 To Studd, '[t]he obvious place to look for an account for what it is to quantify over a domain is to our best semantic theories', and so-called MT-semantics (model-theoretic, i.e. 'cast in set theory') are the natural starting point [‣S19: 61]. MT-semantics has a number of purposes: to account for logical consequence and truth by generalisation over all interpretations; and in linguistics, to 'shed light on natural language quantification' [‣S19: 69]. Studd defines two toy languages, $\mathcal{L}_{SU}$ (the language of set theory) and $\mathcal{L}_{GQ}$ (the language of generalised quantifiers). Since both 'inherit *intended* interpretations from the corresponding fragments of English', we can legitimately ask whether MT-interpretations 'captur[e] the intended interpretations of the first-order language of set theory'.

A similar set-theoretic semantics can be given for 'some', 'most', and so on [following ‣S19: §§ 3.1–2]. Most absolutists, however, accept that there is a straightforward difficulty for MT-absolutism, namely that the availability of a universal set leads to Russell's paradox.[1]

In naïve set theory, there is a universal set. Since the central conceit of dialetheism 'is its ability simply to absorb certain contradictions' [‣P07], it is only natural to first investigate whether dialetheist MT-absolutism is viable, by investigating the viability of paraconsistent naïve set theory. This conforms to the earliest concrete suggestion of a dialetheist defence of absolutism of which I am aware [‣P07].

1. Studd states this more formally but for brevity I omit details [‣S19: § 3.1–2] of what is a fairly uncontroversial conclusion.

> The correct solution here is, I think, a dialetheic one. One of the great strengths of dialetheism is its ability simply to absorb certain contradictions; and, in particular, to provide a simple way of allowing a theory to specify its own semantics. … Paraconsistent set theory quantifies over all sets, and provides a set of all sets to play the role of the domain in a semantic interpretation for the language. All this is entirely natural: no new mechanisms and strategies have to be invoked to handle [absolutely unrestricted quantification].

Such a framework allows familiar set-theoretic reasoning. If, for example, more than half of a choir are men, and all men in the choir wear suits, we can reason that more than half of the choir wear suits, by consideration of the cardinalities of the set of members of the choir, men in the choir, and men wearing suits in the choir.

5.2 *The inclosure schema and principle of uniform solution.* Priest submits that there is already a well-established dialetheist means by which to respond to the sorts of paradoxes absolute generality appears to generate [▸P07].

> Since the contradictions that arise are simple variations of standard paradoxes of self-reference, they all fit the Inclosure Scheme. …Given an interpretation, diagonalisation allows one to construct an object not in its domain (Transcendence); but the object is clearly in the domain of all quantifiable objects, since one quantifies over it (Closure). One should therefore expect inclosure contradictions to arise in this context.

According to the 'principle of uniform solution', the 'same kind of paradox' should be accorded the 'same kind of solution' [▸P00: 123]. This principle is disputed. For example, Badici has argued that Ramsey was right to distinguish logical and semantic paradoxes, and sought to undermine Priest's 'common kind' claim [▸B08]. Landini has questioned whether Priest accurately reconstructed his first example of an instance of the schema—Russell's paradox—and suggested that the schema may not apply to 'paradoxes of "definability"' [▸L09]. Whether or not Priest is right about the inclosure schema is certainly of interest, but I do not propose to settle that question. What is important is that the principle of uniform solution applied to the inclosure schema is one important motive of Priest's suggestion.

I shall instead content myself with showing that the Russell and Williamson–Russell *reductiones* indeed conform to the schema—a fairly routine, if novel, exercise.

5.2.1 *Definition (Inclosure.)* Ω is an inclosure just in case there exist properties ϕ and ψ and a function δ such that:

I $\Omega = \{y : \phi(y)\}$ exists, and $\psi(\Omega)$; and

II for any $x \subseteq \Omega$, if $\psi(x)$, then $\delta(x) \notin x$ and $\delta(x) \in \Omega$.

5.2.2 *Lemma.* A contradiction is derivable from the existence of any inclosure.

Proof. Since $\psi(\Omega)$, $\delta(\Omega) \notin \Omega$. □

5.2.3 *Claim. The Russell* reductio *conforms to the inclosure schema.*

Let ϕ hold just in case x is quantified over absolutely generally, and Ω therefore be an absolutely general domain. Let δ map a domain D to a collection r_D comprising non-self-membered collections in D. Finally, let $\psi(D)$ denote that r_D (not D) is not self-membered.

We now show each requirement. First, by absolute generality, Ω exists. Second, r_D's members are all non-self-membered. If r_D is self-membered, then it is not a member of r_D, so r_D isn't self-membered. Hence $\psi(D)$ in general, and in particular $\psi(\Omega)$.

Third, take any domain (which must be a subset of the absolutely general domain). Then $r_D \notin D$, otherwise $r_D \in r_D$. But at the same time r_D is a perfectly well-defined collection over which the absolutely general domain quantifies, so $r_D \in \Omega$. □

5.2.4 *Claim. The Russell–Williamson* reductio *conforms to the inclosure schema.*

Let $\phi(x)$ hold just in case x is quantified over absolutely generally, and Ω be the absolutely general domain. Then let δ map a domain $D \subseteq \Omega$ to an interpretation i_D on which P applies to all those interpretations in D that do not self-apply P. Finally, let $\psi(D)$ denote of a domain that i_D does not self-apply P.

We now show each requirement. First, by absolute generality, Ω exists. Second, i_D interprets P to apply only to interpretations that don't self-apply P. If i_D self-applies P, it does not self-apply P; therefore, assuming no truth value gaps, i_D does not self-apply P. Therefore, $\psi(D)$. This applies to all D; therefore, in general, $\psi(D)$ may be assumed; and, in particular, $\psi(\Omega)$.

Third, take any domain (which must be a subdomain of the absolutely general domain). Then $i_D \notin D$, otherwise i_D self-applies

P. But at the same time it's a perfectly reasonable interpretation over which to quantify, so $i_D \in \Omega$. □

5.2.5 *Recapture* redux. Both arguments above appear to be proofs by contradiction, which is slightly worrying; indeed, Priest seems to give an exactly parallel proof by contradiction in expounding Russell's paradox [▸P95: 142]. However, by the result given in ▸¶ 9.4.3, for any classical argument $\Gamma \vdash \phi$, there is a paraconsistent argument $\Gamma \Vdash \phi \vee \beta!$ for some β. In this case, ϕ is some dialetheia $\alpha!$, and so $\Gamma \Vdash \alpha! \vee \beta!$, which itself is a contradiction.

5.2.6 One question raised earlier is when we should take a contradiction to be true (▸Q5). The schema gives a partial answer: if accepted, it gives a general class of true contradictions.

5.3 *Recapitulation: the attractions of dialetheist absolutism via a universal set.* We have at least three motives to consider a defence of absolutism via a universal set. The first is the familiarity of set-theoretic semantic theorising. The second is that the principal difficulty for that set-theoretic approach is alleged paradox, and paraconsistent approaches may alleviate that problem. The third is the principle of uniform solution.

5.4 *Expectations of paraconsistent set theory.*

5.4.1 *Non-negotiables.* As stated above, we require a universal set. Non-triviality is non-negotiable. Finally, inferential strength of at least some level is also non-negotiable; a trivial proof theory on which nothing follows from anything else is non-trivial but also insufficient to carry out set theory.[2] Precisely how strong the proof theory and axioms should be is still up for debate.

5.4.2 *Recapture?* Studd warns against 'radical reform [of] mathematical practice…set theory, or model-theoretic semantics' [▸S19: viii]. Dialetheists wary of confrontation could attempt to make 'substantial progress towards recovering classical set theory'.

Priest's ambitions—logical revisionism but recapture of classical mathematics—are not too far from Studd's demands [▸P06: 221]. Most radically, to Routley,[3] classical mathematics should be 'recoverable insofar as it is correct' [▸W21: 98]—and there is no presupposition that it is, indeed, correct.

2. As Incurvati puts it, a set theory should be 'neither weak nor trivial' [▸I20: 103].

3. and Weber, who approvingly cites him.

Therefore, dialetheism about sets leads to rather thorny further questions about attitudes to orthodox set theory and mathematics. On the other hand, non-set-theoretic articulations of absolutism, perhaps using new expressive resources, may not; and their dialetheist variants may not, therefore, face the same questions, as I argue in ‣§ 9.

5.4.3 *Naïveté.* Naïve set theory, of the sort usually abandoned due to Russell's paradox, comprises two fairly intuitive axioms: extensionality $(\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y))$ and naïve comprehension $(\exists y \forall x(x \in y \leftrightarrow \phi(x)))$.[4] Call the result of this NST—naïve set theory.

5.4.4 *To follow.* Even diehard dialetheists admit that current developments in paraconsistent naïve set theory are not too encouraging [‣W22: §§ 2.1–2; ‣P22]. However, no definitive argument against all ways of developing a paraconsistent naïve set theory has, to my knowledge, been developed or formulated. In order to justify my own pessimism about naïve set-theoretic dialetheist absolutism, I shall rather rapidly follow Incurvati's fairly recent survey of the literature [‣I20]. The conclusion I reach, however, is quite provisional.

5.5 *The material and relevant strategies.* For Russell's paradox to avoid triviality, we need to avoid *ex contradictione quodlibet.* (Of course, that is compatible with holding that anything follows from *some* contradictions.) But that is not all.

For example, suppose that $\rightarrow$ is read to validate contraction: $\phi \rightarrow (\phi \rightarrow \psi) \vdash \phi \rightarrow \psi$, and, in addition, *modus ponens.* Then triviality follows [‣I20: 103]. Indeed, *modus ponens* and the axiom $\vdash \phi \wedge (\phi \rightarrow \psi) \rightarrow \psi$ also lead to triviality [‣I20: 104].

The natural question is whether any meaningful set theory at all can be carried out under such constraints. Two strategies naturally present themselves. The *material* strategy takes $\alpha \rightarrow \beta$ to be defined as $\neg\alpha \vee \beta$ and weakens the logic sufficiently that $\{\alpha, \neg\alpha \vee \beta \nvdash \beta\}$. The *relevant* strategy rejects contraction.

5.6 *The material strategy* [‣I20: § 4.3].

5.6.1 NLP combines the axioms of NST with Priest's logic LP. It is a fairly natural starting point. The essential revision is that disjunctive syllogism fails: $\phi, \neg\phi \vee \psi \nvdash \psi$. However, not only is NLP's consequence relation restricted from application of *modus ponens* on pain of triviality; the transitivity of the material conditional and biconditional are

4. Some require that y should not be free in ϕ, whilst others do not; I shall ignore that detail [‣I20: 103].

also ruled out. Consider a valuation $v(\phi) = T, v(\psi) = B, v(\chi) = F$; then $v \vDash \phi \rightarrow \psi, \psi \rightarrow \chi$ but $v \nvDash \phi \rightarrow \chi$ [▸I20: 107]. This rules out rather a lot of classical reasoning. Moreover we cannot prove such claims as the indiscernibility of identicals: $\phi(x), x = y \nvdash \phi(y)$. A dialetheist could claim that, in a contest of intuitions, naïve set-theoretic axioms should take priority over the indiscernibility of identicals, but such a claim seems tenuous at best [▸I20: 109]. Priest has little hope for NLP, and I am not one to doubt him [▸P06: 250].

5.6.2 *Minimal inconsistency* [▸I20: 109–10]. Priest has proposed LPM as the background logic. Its consequence relation is stronger, and,

> [i]n particular, every classical consequence of a consistent set of premisses is an LPM consequence, since in consistent situations the minimally inconsistent models are just the classical models.

The result is NLPM (naïve set theory with LPM). Unfortunately,

> Thomas [▸T14: § 4] has shown that NLPM's only model is one in which $\forall x \forall y(x \in y \wedge x \notin y)$ holds and identity is standard.... Moreover, NLPM has only one model up to isomorphism, namely a one-element model.

Various intermediate options have also been explored [▸I20: 111].

> In particular, three alternative versions of LPM have been developed by Marcel Crabbé [▸C11]. Using one of them as background logic, the situation is exactly the same as for NLPM... Using the other two alternative versions of LPM as background logic we avoid almost triviality but at the price of not improving upon $\mathrm{LP}_=$ as the background logic: the resulting naïve set theories cannot prove the existence of sets that behave like singleton sets, sets that behave like ordered pairs, and sets that behave like infinitely ascending linear orders.

Accordingly, I provisionally conclude that Incurvati is right: 'the prospects for... the material strategy look rather dim, and those for developing it on the basis of LP and cognate systems even dimmer' [▸I20: 111].

5.7 *The relevant strategy* [▸I20: § 4.4] takes the conditionals in the axiomatisation of NST to validate *modus ponens*. There is a candidate non-trivial theory in which a reasonable number of constructions familiar from standard set theory can be carried out: Weber's NDLQ [▸W12]. The central problem is that the path to the underlying logic, DLQ, appears tortured and *ad hoc* [▸I20: 117].

> Obviously the Counterexample Rule $[\phi, \neg\psi \vdash \neg(\phi \rightarrow \psi)]$ appears *prima facie plausible*... But then, disjunctive syllogism also appears prima facie plausible, but almost no paraconsistent logic can include it, on pain of triviality.

Weber argues that 'we want the strongest [in terms of deductive strength] logic possible that does not explode when given a comprehension principle' [▸W12: 73]. As Incurvati points out, however, some dialetheist strategies are both deductively incommensurable (each has deductive consequences absent in the other) and incompatible (on pain of triviality). There is no obvious reason to prefer one to the other. Finally, 'current attempts to provide it with a genuine principle of extensionality fail, so that it cannot be regarded as a *set* theory' [▸I20: 121].

Moreover, to the extent that a fuller programme of mathematics grounded in NST in NDLQ can be worked out,[5] it is unclear what to make of it. For example, indispensability arguments[6] do not appear to apply to abstract objects as conceived of in the resulting revisionist mathematics, since natural science appeals to orthodox, not revisionist, mathematics. Perhaps that is simply an unfair artefact of history—but we can have little faith in that defence of revisionist mathematics unless a substantial programme of natural science is carried out using revisionist mathematics, which prospect, alas, remains distant.

5.8 *The model-theoretic strategy* [▸I20: § 4.5]. A third strategy is to argue that 'in a context where no sentence is true and false, [dialetheists] may make use of classical logic'; therefore, 'the dialetheist can also restrict attention to a consistent subdomain of the naïve universe of sets, and use classical logic in deriving things from axioms describing that subdomain' [▸I20: 121].

One problem is that we must either accept only 'a proper segment of [a] given model of ZF' or identity that behaves 'non-standardly' [▸I20: 124]—neither of which is particularly desirable. Another is that the strategy delivers merely 'a particular interpretation of NST which contains a model of ZFC', which 'does not tell us that the true interpretation of set theory enjoys this property' [▸M15: 182]; so far, the only reason to believe that this is so is that it is 'an appealing picture' [▸P06: 257]. There remains much work to do in following this strategy.

5.9 *Conclusion.* Whether or not all happy set theories are alike,[7] each paraconsistent set theory we have encountered has been unhappy in its own way. Two particularly common difficulties arise: triviality (or near-triviality); and adhocness. Studd's verdict, then, seems to be right. However, naïveté has been assumed throughout. That is natural—a defence of absolutism via a universal set can hardly

5. For reasons for optimism, see [▸W21].
6. See e.g. [▸C01].
7. Hamkins proposes that there is a 'set-theoretic multiverse' [▸H12].

proceed without one. In classical logic, naïveté cannot be wholly accepted—a universal set cannot coëxist with the usual form of comprehension. I have not yet explored the prospects of paraconsistent set theories much closer to existing practice, because the form of absolutism under discussion requires a universal set. Other forms of absolutism, however, impose no such requirement; I therefore return to the prospects of non-naïve paraconsistent set theory in ▸§ 9.

6 Plural dialetheist absolutism.

Plural dialetheist absolutism has one major advantage: it overcomes certain *expressibility deficits* that 'traditional pluralists' must face. There is another way of avoiding those expressibility deficits, namely Florio and Linnebo's 'critical absolutism'. I argue that the critical absolutist's articulation of absolutism is worryingly circular.

Plural dialetheist absolutism faces two charges. The first is that it may be unintelligible, along with plural absolutism; I do not think this objection is particularly decisive. The second is that, in order to simultaneously avoid lapsing into dialetheist absolutism via a universal set and maintain the motive for dialetheism, it must take an *ad hoc* view of so-called 'universal singularisations'. Nevertheless, it is viable, and its virtues inform the form of dialetheist absolutism I shall advocate in chapter ▸III.

6.1 *Plural absolutism: the very idea.* Talk of plurals is elliptical. A plurality comprises one or more things. Pluralities are not to be taken as sets; rather, 'the loose 'plurality'-talk should be taken as elliptical for a paraphrase given in the plural metalanguage' [▸S19: 73]. Accordingly, in discussing a plurality, we do not commit ourselves to the existence of some object of which the one or more things for which it is elliptical are all member. The underlying plural metalanguage is fairly comprehensible as a slight enrichment of natural language. We index pronouns with variables (e.g. it_v) in order to disambiguate [▸S19: 247].

> The interpretation of an $\mathcal{L}_{\text{PSU}}$ formula ϕ is given by its English translation $(\phi)^{\text{tr}}$, where
>
> $(v \prec vv)^{\text{tr}}$ = it_v is one of them_{vv} [..., and]
> $(\forall vv\phi)^{\text{tr}}$ = any one or more things_{vv} are such that $(\phi)^{\text{tr}}$ and $(\phi^*)^{\text{tr}}$[.]
>
> The additional conjunct employs Boolos's trick of simulating the assignment of 'the empty plurality' to vv by taking ϕ^* to be the result of replacing each occurrence of $u \prec vv$ in *phi* with $u \neq u$.

All talk of pluralities is straightforwardly regimented into $\mathcal{L}_{\text{PSU}}$ and thence into the translation in augmented English, which, in turn, is perfectly intelligible, if rather strained.

Plural absolutists argue that

▸Q4 'the needs of quantification are served simply by there being the items quantified over' [▸S19: 73].

In other words, quantified statements are statements about some plurality—about one or more things.

Studd's also asks whether the needs of our 'best semantic theories' are also so served. Recall that MT-semantics interpreted quantifiers as ranging over set-domains. P-semantics, by contrast, 'specif[ies] a plurality to serve as the universe, and pluralities as extensions for ß [a predicate applying to sets] and ∈' [▸S19: 73].

Returning to $\mathcal{L}_{SU}$, matters seem fairly straightforward. Studd shows that some plurality-domain comprises everything, some plurality-extension comprises every set, and some plurality-extension comprises every element-set pair. Studd, however, appears to suggest that the interpretation of $\mathcal{L}_{GQ}$ must employ '"superplural" resources' [▸S19: 76].[8] We can first state how, in terms of these superpluralities, $\mathcal{L}_{GQ}$ is interpreted. A plural metalanguage is enriched with plural variables $xx, yy, \ldots$; we now enrich it in turn with superplural variables xxx, yyy and superplural quantifiers $\forall xxx, \forall yyy$, as well as an overloaded plural-superplural predicate $\prec$. Then a rough sketch of the approach is as follows [▸S19: 76]:

> [T]he set-universe is replaced with a plurality-universe, mm, and each set-predicate-extension is replaced with a plurality-predicate extension based on mm (i.e. a plurality of members of mm). ... Under an SP-interpretation, a quantifier-extension based on mm is a superplurality comprising plurality-predicate extensions based on mm. ... Adapting the standard set-theoretic tricks for encoding functions, an SP-interpretation takes the extension of each determiner to be a superplurality-encoded function, mapping each plurality-predicate-extension based on mm to a superplurality-quantifier-extension based on mm.

6.2 *Intelligibility.* How are we to construe talk of superpluralities? These appear more problematic than pluralities. We begin with the not wholly implausible premiss that the intelligibility of superplurals is dependent on the existence of suitable higher-order quantification in natural language; we could then claim that such quantification is unavailable. In answer to the first premiss, we might wonder why

8. There is nothing wrong with developing the superplural approach, but it is natural to wonder whether that is the only option. It turns out that not just superplurals but arbitrary higher-order plurals are necessary in order to give a so-called 'generalised semantic theory', so even if Studd's approach needlessly appeals to superpluralities, there is good reason to think that they are unavoidable.

natural language 'already encompasses every intelligible kind of expression'; for some expressions are made intelligible 'by learning how to use them in the right sort of way' [▸S19: 79]. The intelligibility of the metalanguage is non-negotiable, then, but it does not entail that the relevant sorts of quantification must already be present in natural language. The second premiss is also disputed [▸G21] but I do not propose to enter that debate; instead, I accept a disjunctive criterion of intelligibility: either the relevant sorts of quantification should be shown to be paraphrases of equivalents in natural language, or some account of their use should be provided.

6.3 *Dialetheism, plural Cantor, and a trilemma.* It is possible to prove a plural analogue of Cantor's theorem for sets [▸F+21: § 11.2].

> *Plural Cantor.* For any plurality xx with two or more members, the subpluralities of xx are strictly more numerous than the members of xx.
>
> … Let a *singularization* be an injective mapping from the subpluralities of some objects xx into objects.

This leads to a trilemma [▸F+21: § 11.3].

FIRST HORN

Universal singularizations are impossible.

SECOND HORN

It is impossible to quantify over absolutely everything.

THIRD HORN

There is no plurality that is universal or all-encompassing.

Suppose that absolutely general quantification is possible and there is a universal plurality available. A universal singularisation would injectively map from subpluralities of the universal plurality to some objects. But that contradicts plural Cantor.

This looks like it might be fertile ground for the dialetheist. Although Weber has argued paraconsistently for Cantor's theorem [▸W12], Peterson shows 'by similar methods…that no non-empty set satisfies Cantor's theorem' [▸P23]; and such a result perhaps could be shown in the theory of plurals too. So we might be able to avoid all three horns paraconsistently.

Thus, Priest proposes that ‘[y]ou can quantify over all objects and have arbitrary pluralities of objects’ [▸P22]—in other words, we reject the second and third horns. But what about the first?

6.3.1 *Traditional absolutism* accepts the first horn. In this case, plural Cantor does not pose a problem, and there is no obvious motive for dialetheism. This is orthodox e.g. in set theory.

Unfortunately, traditional absolutists appear to be committed to a hierarchy of higher-order plural or typed resources, due to a result following from plural Cantor [▸F+21: 11.A]. Acceptance of the underlying results appears to be fairly widespread but it is stated diffusely—sometimes type-theoretically [▸L+12] and sometimes in terms of orders of plurality [▸F+21: § 11.5]; but the result retains a fairly similar structure in each case.

The central premiss of the argument is that, for any particular language $\mathcal{L}$, we should expect a *generalised semantic theory* of a given language: that is, a theory ‘of all possible interpretations the language might take’ [▸L+12: 275]. Where quantifiers are suitably restricted this would amount to a model theory; but we have moved on from absolutism via a universal set. The central reason to seek such a theory is that it is natural to define logical consequence in terms of interpretations [▸W03: 425].

> Sooner or later the naive theorist will want to generalize over all (legitimate) interpretations of various forms in the language. For example, the inference from $\forall x Px$ and $\forall x(Px \rightarrow Qx)$ to $\forall x Qx$ is truth-preserving however one interprets the predicate letters P and Q. Such generalizations are the basis of Tarski's account of logical consequence and its model-theoretic descendants.

In speaking of *order*, I refer to the type restrictions that both pluralists and other type theorists endorse in their accounts of quantification: a plural variable cannot be quantified over by a singular quantifier, and a type 3 variable cannot be bound by a type 5 quantifier (for example); nor can we permissibly substitute constants of the wrong type. The following result applies generally to typed languages.[9]

> ASCENT THEOREM (ARBITRARY FINITE FORM)
> Assume traditional plural logic and the possibility of absolute generality at every finite order n. Then a generalized semantics for a language of order n cannot be given in another language of order n but can be given in a language of order $n + 1$.

9. Florio gives a roughly parallel result [▸F14: § 4.1].

As Florio and Linnebo argue, this leads to an expressibility problem for three reasons [▸F⁺21: § 11.6].

I ‘Type-unrestricted generality appears to be possible. For example, it appears meaningful to ask whether the law of extensionality holds at every order of the type-theoretic hierarchy. ... There is no such thing as quantification across all orders at once.’

II Type-unrestricted generality is theoretically valuable—for example, in stating plural Cantor above. Type-theoretic restrictions preclude ‘proper express[ion] and discuss[ion]’ of these questions.

III It is unclear how the type theorist can even articulate their view that there is a hierarchy ‘without a top level’: to ‘state that quantification of every order can be extended... we need to generalize across all the orders’.

To overcome these expressibility deficits it is natural to seek to remove the type restrictions. Classically, this commits us to the third horn [▸F⁺21: § 11.7], considered below.

6.3.2 Reject the first horn, accept singularisation by set-formation. This is Priest's official view.[10] The problem is that we must then develop a paraconsistent naïve set theory, which task I argued at least remains to be satisfactorily completed in chapter ▸II.

6.3.3 Reject the first horn, and reject singularisation by set-formation. Priest appears to agree that prospects for a paraconsistent approach via set theory appear to be poor: for he proposes that a virtue of a paraconsistent plural logic, as opposed to a paraconsistent set theory, is precisely that there are far fewer demands on a theory of pluralities than there are on set theory [▸P22]. There is no established body of *plural* practice to recover in the same way that there is vis à vis set theory, and little (despite the best efforts of some) has been built on it, compared to set theory.

Such a position will invariably face charges of adhocness. This is for two reasons. The first is that we must identify a candidate means of universal singularisation, and simultaneously maintain that set-formation does not universally singularise. This is not wholly impossible; the grounds for rejecting universal singularisation by sets do not impugn the grounds for accepting universal singularisation, for example, through interpretations: given any one or more things, there is, surely, an interpretation that applies P to exactly those things.

10. Personal correspondence.

The second is that we must then explain why some pluralities correspond to sets but others do not. Two possible explanations are that '(i) the uncollectable pluralities are uncollectable because their members are too numerous; or (ii) because there's no stage in the iterative hierarchy when all their members are available' [▸S19: 191]. In answer to the first, Linnebo asks '[w]hy should this particular cardinality mark the threshold? Why not some other cardinality?' [▸S19: 192]. In answer to the second, Studd objects that '[t]he availability of the finite ordinals at some stage in the process of set formation can't be the real reason that they are collectable because, in reality, there isn't any such process. Instead, the metaphorical explanation is at best a colourful way of presenting a more sober explanation', namely that the finite ordinals are collectable 'because there is some ordinal that exceeds each of their ranks', and, since there is no such ordinal in the case of the finite and transfinite ordinals, the latter are uncollectable. This, in turn, faces a similar difficulty to the limitation of size explanation. So 'these two collectedness theses' indicate 'precisely where the line falls. But neither succeeds in explaining why it falls where it does.'

Arguably, these questions arise for anybody who privileges ZFC, and not just in the context of the debate over absolute generality or plural absolutism. In ▸¶ 9.3, I shall follow e.g. Maddy in emphasising *extrinsic* justifications of set theory [▸M11]. These could answer the query above.

By way of context, two non-dialetheist options fall into this way of framing the debate.

6.3.4 *Relativism* accepts the second horn.

6.3.5 *Critical absolutism*—proposed by Florio and Linnebo—accepts the third horn. The removal of these type-theoretic restrictions leads to the unpalatable conclusion that there is no universal plurality [▸F+21: § 11.7]; but Florio and Linnebo insist that absolutely general quantification remains possible. This raises the question: what, exactly, is absolutist about the resulting position, if it cannot appeal to a universal plurality? Just as the pluralist disassociated sets and domains, so must the critical pluralist disassociate pluralities and domains. For there is no universal plurality to be associated with an absolutely general quantifier on this view. Indeed, Florio and Linnebo develop a theory of plurals largely in parallel to orthodox set theory.

The principal difficulty I see with this proposal is that Florio and Linnebo do not really characterise in what exactly absolutely general quantification is to consist in the absence of a universal plurality, or, indeed, what quantification is to consist in in the absence of plurals. Of course, they *say* that they are absolutists [▸F^{+}21: § 11.1], and they do not appear to have any particularly relativist commitments, but neither alone justifies confidence that their position is absolutist, let alone that their position presents an attractive defence of absolutism. I discuss difficulties in this vein in ▸§ 7.

6.4 *Recapitulation.* The most promising dialetheist plural absolutist position takes plural Cantor's theorem to be a dialetheia but rejects naïve set theory. It avoids expressibility problems, but is doubly *ad hoc*—in respect of candidate universal singularisations (interpretations, not sets), and in respect of which pluralities are coextensive with sets.

The options explored so far have all been subsentential (▸§ 4). In the next chapter, I shall discuss sentential absolutism (and dialetheist sentential absolutism), and in ▸§ 10, I shall assess the relative merits of my preferred form of dialetheist plural absolutism and my preferred form of dialetheist sentential absolutism.

III Sentential dialetheist absolutism.

7 Sentential absolutism.

7.1 *On articulating sentential absolutism.* So far, we have tried to associate quantifiers with sets, and with pluralities. The second attempt founders less badly than the first, but still incurs a seemingly substantial adhocness, avoidable only by returning to the project of paraconsistent naïve set theory. It is now time to explore the sentential option.

The first task is to articulate absolutism in terms of sentential semantic facts. I shall consider two types of sentential semantic fact: quasi-homophonic truth conditions, and inferential commitments.

What precisely do I mean by articulating absolutism in terms of sentential semantic facts? It might be thought that straightforward assertions that some discourse concerns absolutely everything and that the sentences therein are absolutely generally quantified would suffice (in the vein of ‣¶ 1.1). The problem is that the appeal to 'absolutely everything' is circular; relativists could claim that it fails (perhaps on pain of triviality.) The absolutist could also claim that, in an absolutely general context, 'absolutely nothing is excluded as irrelevant' [‣W03: 415]; but absolutely everything and nothing appear to be interdefinable, and it is not clear that any progress is made. Accordingly, absolutists should be able to state or at least explain the semantic fact of the absolute generality of some quantified sentences in terms of other semantic facts.

In discussing subsentential absolutism, as I have called it, I assumed that subsentential absolutism was intelligibly and meaningfully absolutist: that naïve set-theoretic dialetheist absolutism was satisfactory *as an articulation of absolutism*. The question was not whether naïve set-theoretic dialetheist absolutists really were absolutists, but, rather, whether they could even formulate a viable position. The absolute generality of putatively absolutely general resources in the metalanguage was obvious; the charge was that a metalanguage in which such resources are available could only avoid triviality at undesirable cost. In these metalanguages, it is quite easy to characterise absolute generality by appeal to other (subsentential) semantic facts. Here are two examples. The first is that it is possible to *determine* when a sentence is absolutely generally quantified on the basis of the semantic facts. For example, the absolutist typically claims that ▸(10) quantifies absolutely generally and that ▸(11) does not.

(10) Everything is mereologically simple.

(11) Everything is in the suitcase.[1]

Second, we might also ask what *explains* ▸(10)'s absolute generality, what it *means for* ▸(10) to be absolutely general, what it *is* for ▸(10) to be absolutely general, what exactly is absolutely general about absolutely generally quantified sentences, or how to *define* (in terms of other semantic facts) the claim that ▸(10) is absolutely general. Sometimes, such queries are misplaced: some facts appear to be primitive in ways that preclude further explanation, definition, or articulation of their meaning. To dismiss these queries in respect of absolute generality, however, would be too hasty. What it means to quantify absolutely generally could have a decisive effect on whether absolutely general quantification is possible; ideally, we should not take claims that are contested so to be explanatorily primitive. A naïve set-theoretic dialetheist (subsentential) absolutist, for example, identifies a set corresponding to each quantifier. We could say that the universality of the universal set explains ▸(10)'s absolute generality, that what it means for ▸(10) to be absolutely general is that the set associated with its quantifier is universal, that absolute generality is to be defined by a quantifier's being over a universal set, and that what is absolutely general about ▸(10) is that its quantifier is associated with an absolutely general set.

The two demands should be distinguished. The relativist need not object to the drawing of *some* distinction between *putatively* absolutely generally quantified sentences and other quantified sentences. It could simply be that they seek to draw that distinction without drawing on absolutely general resources—perhaps in the metametalanguage. Perhaps the absolutist can identify a category of uses that intuitively should be given an absolutely general reading: perhaps they can argue that there is some intuitive sense in which e.g. theologians and ontologists will find themselves speaking 'absolutely generally' whilst ordinary people speak more prosaically with contextual restrictions

1. Contextual domain restriction may well operate pragmatically [▸s$^+$00: § 6] but I shall assume that some sort of truth-conditions are obvious in this case, whatever the mechanisms by which they are generally settled.

particular to their discourse; but the distinctness of putatively absolutely general uses does not guarantee that they are, in fact, absolutely general. Merely identifying putatively absolutely general quantification does not appear to be very useful in ascertaining what exactly it means to quantify absolutely generally. Suppose, for example, that there were a language in which all contextual domain restrictions were explicitly articulated and marked. 'Everything in the suitcase' would, unmarked, mean (and be understood to mean) that the Radcliffe Camera and I are also in the suitcase. It might be informative to say that absolutely general quantification is simply contextually unmarked quantification, but not in answer to the question of what it means to absolutely generally quantify.

Turning to a sentential approach is partly motivated by the undesirable effects of the ontological commitments of subsentential articulations of absolutism. Accordingly, the metalanguage should not, ideally, incur substantial ontological commitments in the same way that set theory did.[2] This appears, correspondingly, to limit the resources available in answering such queries, and for this reason articulating absolutism in terms of sentential semantic facts is surprisingly difficult.

7.2 *Quasi-homophonic truth conditions.* First, I shall consider a suggestion by Studd: the absolutist can furnish truth conditions quasi-homophonically, avoiding potentially problematic ontological commitments [‣s19: 79].

> [A] Quinean absolutist has an alternative means at his disposal to capture the intended interpretations of $\mathcal{L}_{\text{SU}}$ and $\mathcal{L}_{\text{GQ}}$. He can avoid reifying semantic values, or universes, by stating semantic theories in the less ontologically committed quasi-homophonic style, primarily associated with Davidson's meaning-theoretic appropriation of Tarski-style truth theories.

On this proposal, we can define truth under an interpretation according to the following.

T*-ß ßv is true$_\sigma$ iff $\sigma(v)$ is a set.

T*-∈ $u \in v$ is true$_\sigma$ iff $\sigma(u)$ is an element of $\sigma(v)$.

T*-= $u = v$ is true$_\sigma$ iff $\sigma(u) = \sigma(v)$.

T*-$\forall v$ $\forall v \phi$ is true$_\sigma$ iff everything a is such that ϕ is true$_{\sigma[v/a]}$.

2. I take no position on whether plural logic is ontologically committed in the same way *simpliciter*, but, so far as absolute generality is concerned, it appears to incur similar commitments via singularisations on pain of adhocness.

> These clauses serve to co-ordinate the truth-conditions for object language formulas containing ß, ∊, and $\forall v$ with those of the corresponding metalanguage sentences containing 'set', 'element', and 'everything', without the need to reify semantic values as sets or pluralities. In particular, provided the absolutist's metalanguage use of 'everything' ranges over absolutely everything, the clause T*-$\forall v$ ensures that $\forall v$ universally quantifies over absolutely everything. In doing so, however, the theory carries no ontological commitment to set-domains or -semantic values and no ideological commitment to plural resources or similar.

The absolutist can distinguish ▸(10) and ▸(11) by pointing out that a clause along the lines of T*-$\forall v$ gives the truth condition of ▸(10), but not of ▸(11); some other clause, taking into account contextual domain restriction, performs the latter task. Quasi-homophonic truth conditions, therefore, may well suffice in articulating *when* absolutely general quantification is present. This is unsurprising. All that has been achieved quasi-homophonically is that *putatively* absolutely generally quantified sentences are paired between object- and metalanguage; and, as noted above, this is not obviously objectionable to the relativist.

Let us turn to the second task: elucidation of absolutely general quantification on its own terms, rather than the distinction between it and restricted quantification. As an account of absolutely general quantification *per se*, how helpful is T*-$\forall v$? By comparison with appeals to universal sets and pluralities, it seems quite disappointing and thin. It might seem that the latter is a virtue: if the whole point of the quasi-homophonic approach to truth conditions is to avoid problematic ontological commitments, it is hardly reasonable to then turn on the approach for failing to have them. Nor would it make much sense to complain, for example, about

T*-∧ $\phi \wedge \psi$ is true$_\sigma$ iff ϕ is true$_\sigma$ and ψ is true$_\sigma$.

That example, however, illustrates a contrast between conjunction and quantification. Conjunction is generally considered unproblematic, in the sense that, first, it is generally thought that examples of conjunction in natural language behave roughly as they do commonsensically, and, second, that their so behaving is not at risk of leading to triviality. Absolutely general quantification, however, is problematic, in the sense that, first, it is disputed that it behaves roughly as it should commonsensically, and, second, its so behaving would, according to some, lead to triviality. It is unsatisfactory for the absolutist to simply 'co-ordinate truth conditions': the availability of absolutely generally quantified sentences in the metalanguage to 'co-ordinate' with those in the object language is at stake.

It is understandable that Studd does not explore this difficulty: to Studd, this view is fairly quickly refuted as contradictory, and so, since he is classically minded about logic, it is unlikely to him that a viable view could be developed on such a premiss. I return to that argument in ▸§ 8. But a sentential absolutist seeking to articulate their sentential absolutism truth-conditionally cannot avoid this problem.

This is not to say that quasi-homophonic or more broadly Davidsonian semantics is uninformative generally; indeed, closer examination of how it could be informative suggests that the negative conclusion above is unsurprising. Davidsonian semantics formally regiments the sentences of natural (or toy) languages in order to compositionally give their meanings. It matches sentences of the object language to sentences in the metalanguage that are 'alike in meaning' [▸L+07: 28]. It does so compositionally, in the following sense [▸L+07: 18]:

CM A compositional meaning theory for a language $\mathcal{L}$ is a formal theory that enables anyone who understands the language in which the theory is stated to understand the primitive expressions of $\mathcal{L}$ and the complex expressions of $\mathcal{L}$ on the basis of understanding the primitive ones.

Such a theory may clarify the structural position of absolutely general quantification relative to other features of language, such as connectives. It may allow us to regiment into some appropriate logical form sentences of natural language (including those that putatively quantify absolutely generally). However, what it does not do is avoid circularity in giving the meaning of absolutely generally quantified sentences: it is their very intelligibility—in any language, metalanguage or object language—that is at stake in this debate. The truth conditions of absolutely generally quantified sentences have no way of ensuring that, for example, the quantifiers of the metalanguage are not covertly restricted after all.

Moreover, even if it can be assumed that we understand what absolutely generally quantified sentences in the metalanguage mean, and that, in the Davidsonian vein, we properly understand their logical form, the relativist could still have their revenge by showing the result to be trivial. The regimentation of an object language into a suitably structured metalanguage must therefore be supplemented with a logic, rather than just a logical *form*. At that point, however, it is unclear why exactly the regimentation was needed in the first place:

what was important was accounting for the patterns of inference that absolutely general quantification must license. And there is not too much mystery in the logical form of *absolutely general* quantification, whatever work there may be of interest concerns generalised quantifiers, contextual domain restriction, and so on.

Truth conditions, however, do not exhaust the options of the sentential absolutist. For they are only one kind of sentential fact—and a rather special one. They do not, for example, associate sentences of the object language with one another; rather, they pair object- and metalanguage sentences. It is natural, therefore, to consider whether sentential semantic facts relating different object language sentences could furnish a satisfactory and less circular account of what it means to quantify absolutely generally. I shall now turn to the *inferential commitments* of putatively absolutely generally quantified sentences.

7.3 *Inferential commitments.* There is a distinctly absolutist position that can be detected in the exchange over the Williamson–Russell *reductio* (‣¶ 2.2.5). The relativist claims that the quantifier was restricted to avoid ranging over the constructed interpretation $I(R)$. This is not an option for the absolutist. One option is to simply deny that there are such things as the relevant interpretations. Interpretations were motivated by the need to give a 'Tarskian definition of logical consequence'; and Williamson argues that we do not need to resort to interpretations to so define consequence [‣W03: 452]. (The trade-off incurred is essentially the same as that higher-order plurals or type-theoretic restrictions must accept, as discussed in ‣§ 6.) The dialetheist response is simply to accept that there are such interpretations and the corresponding contradiction. The general lesson is that the absolutist, whenever confronted by a putative counterexample to their absolutely general quantifier, must either show that there is no such thing as the putative counterexample in the first place, or somehow show that their quantifier acceptably ranged over it; they cannot accept that there was such a thing as the counterexample and that their candidate absolutely general quantifier did not range over it. This suggests that there may be room for another way of stating the sentential semantic facts in virtue of which a sentence quantifies absolutely generally—by appeal to their inferential commitments.

What are the commitments that characterise absolutely general quantification? The obvious candidate is the standard natural deduction rules for the universal and existential quantifiers are absolutely general; they do not appear to involve any restriction on what value the quantified variable takes. A first proposal is that

‣Q4 the meaning of quantified sentences is given by their inferential commitments, such as universal instantiation and generalisation. Absolutely generally quantified sentences' absoluteness arises from the generality of the proof rules of a quantifier.

Yet arguments against absolutism do not appear to locate the problem in first-order logic; relativists too make use of them. To conform to these rules is plausibly necessary, but hardly sufficient. I shall now outline three proposals.

7.3.1 Absolutism as anti-relativism. Relativism plausibly faces an objection from ineffability,[3] but such objections may not be decisive. Suppose that the relativist can coherently articulate their view. Given a candidate absolutely general quantified sentence $\forall x \phi(x)$, the relativist claims that there is something y that was not quantified over. If the absolutist insists, further, that $\phi(y)$ follows, the relativist shows that contradiction results. The absolutist must either deny that there is such a thing, show that contradiction does not follow, or accept the contradiction. (To be precise, the absolutist must accept that, from a candidate absolutely generally quantified sentence $\forall x \phi(x)$, $\phi(x)$ follows, whatever x may be, provided it indeed exists.) Absolutists, therefore, could hold that

‣Q4 Absolutely generally quantified sentences' absoluteness arises from the generality of their proof rules—in particular, in application even to putative counterexamples to their absolute generality, provided that they exist.

On its own terms, the proposal is *prima facie* tenable. The standard mechanism for generating a candidate counterexample is some sort of diagonalisation that shows a concept to be indefinitely extensible. The dialetheist accepts the dialetheia implied by the diagonalisation but denies that triviality follows. If objections from indefinite extensibility are the primary motive for relativism (‣Q2), the absolutist as anti-relativist is able to achieve a strong position, provided that worries about logical or mathematical revisionism elsewhere can be addressed. If the relativist cannot even articulate something over which candidate absolutely general quantifiers do not range, it is unclear what motivates their position. Even saying 'there is something that is a counterexample to your theory to which I am unable to refer' appears to allow the absolutist to insist that universal generalisations apply to whatever that something is too.

3. See ‣¶ 1.2 for discussion.

Returning to the first task above, then, the absolutist as anti-relativist holds that ▸(10) is absolutely general because any instance inferred from ▸(10) 'x is mereologically simple' must be accepted as following, including, in particular, those thought to be counter-examples. Admittedly claims such as ▸(10) are not usually those from which relativists seek to derive contradiction. But even on these limited resources contradictions can be derived, as in ▸§ 8.

It could be argued—although I am not wholly sure—that the only problem with circular characterisations of absolute generality is that it is unclear what to make of them in potentially paradoxical or problematic situations—so, once those problematic cases have been properly characterised, there is no more problematic circularity. If that is so, absolutism as anti-relativism seems to be fairly satisfactory.

Nevertheless, the suspicion may remain that quantification over putative counterexamples to the unrestrictedness of their candidate quantifier falls short of the sort of unrestrictedness that would be desirable. At the very least, what has been ensured is that anti-relativist absolutism cannot be accused of covert relativism: it ensures that there is a disagreement with the relativist.

7.3.2 Absolutism over languages. I shall now expound a proposal of Warren's, intended to elucidate the notion of indefinite extensibility. Ultimately we only need some parts of it, but I shall sketch a little more for context.

Recall that Warren distinguishes two broad traditions in semantics [▸W17: 87].

> **Bottom Up:** Subsentential semantic facts are explanatorily prior to sentential semantic facts (ceteris paribus).
>
> **Top Down:** Sentential semantic facts are explanatorily prior to subsentential semantic facts (ceteris paribus).

Warren regards use 'use theories of meaning as endorsed by Wittgenstein…normative inferentialist theories as endorsed by Sellars [and] Brandom[, and] inferential/conceptual role theories as endorsed by Block, Harman, [and] Field' as *top down*; on such a view, 'facts about a quantifier's domain are explained by facts about the semantic properties of the whole sentences involving the quantifier'.

> **Quantifier Inferentialism:** Subsentential expression Q in language $\mathcal{L}$ is an (unrestricted) type i quantifier expression just in case Q plays, in $\mathcal{L}$, the inferential role of an (unrestricted) type i quantifier expression.

i is meant to range over different kinds of quantifier (universal, existential, …) and has nothing to do with type hierarchies.[4] As Warren points out, quantifier inferentialism so defined merely determines when a subsentential expression '*counts as*' the relevant sort of quantifier [‣w17: 92]. According to 'what is commonly called inferentialism or conceptual role semantics', the inference rules governing a quantifier 'determin[e its] meaning'; and, indeed, one could 'do away with truth conditions and standard semantics altogether'. Inferentialism and Warren's quantifier inferentialism can but need not be combined; I do not commit myself to such a view.

Warren then calls the combination of quantifier inferentialism and top down semantics 'quantifier deflationism'. He proceeds to characterise the position that results from the now familiar relativist move of attempting to furnish something not quantified over by the absolutist. Warren makes the point about sets, but the point could equally apply to interpretations; and I am surer of their indefinite extensibility than that of sets for the reasons outlined in ‣§ 5.

Suppose that we attempt to 'form a conception of the set of all non-self-membered sets' [‣w17: 102]. In doing so, we might introduce a term for the Russell set r, and so we move from some original language $\mathcal{L}$ to an expanded language $\mathcal{L}^+$. The relativist will then claim that the meaning of $\forall$, which is common to the signatures of $\mathcal{L}$ and $\mathcal{L}^+$, differs between them: *i.e.*, they cannot be translated into each other. This is justified by two claims. First, $\forall$ in $\mathcal{L}$ has the same *inferential rôle* as $\forall$ in $\mathcal{L}^+$—that is, the same proof rules apply, and 'by hypothesis, neither is restricted'. Second, by Russell's paradox, $\forall$ in $\mathcal{L}$ is not extensionally equivalent to $\forall$ in $\mathcal{L}^+$, so has a different meaning. The result of such moves is generally characterised thus.

> **Quantifier Pluralism:** There are languages $\mathcal{L}$ and $\mathcal{K}$ with expressions $Q_\mathcal{L}$ and $Q_\mathcal{K}$ respectively, such that (1) $Q_\mathcal{L}$ and $Q_\mathcal{K}$ are both unrestricted quantifier expressions of the same type and (2) $Q_\mathcal{L}$ and $Q_\mathcal{K}$ mean different things; that is, $Q_\mathcal{L}$ and $Q_\mathcal{K}$ cannot be translated into each other.

That $\forall$ varies in usage is, to Warren, an *explicans* rather than *explicandum*, following the top-down view of semantics. 'The facts about how are usage changes are, according to quantifier deflationism, freestanding, explanatorily speaking'; and they, in turn, explain the difference in meaning between the subsentential components of the signature given by the quantifier.

4. Personal correspondence.

How is this account helpful? It allows us to distinguish unrestrictedness and absoluteness. Unrestrictedness is language-relative: a quantifier Q in a language $\mathcal{L}$ is *unrestricted* just in case there is no other quantifier in $\mathcal{L}$ of which Q is a restriction. A quantifier Q is *absolutely unrestricted* if there is no language $\mathcal{K}^+$ 'from whose perspective Q is restricted'. The quantifier $\forall x$ in $\langle \mathcal{N}, 0, 1, +, < \rangle$ is unrestricted. But is it absolutely unrestricted? Presumably not: it cannot interpret, for example, ZFC. *Sentential absolutism* therefore can be formulated as the claim that there is an *absolutely unrestricted* quantifier and that its availability does not lead to triviality.

The languages in question arise from the usual mechanisms by which relativists seek to expand the domain. For example, given a language $\mathcal{L}$ in which we conduct set theory, we can express naïve comprehension in an expanded language $\mathcal{L}^+$ by

NC+ $\exists^+ y \forall x (x \in y \leftrightarrow \phi(x))$

where $\exists^+$ is unrestricted in $\mathcal{L}^+$ and $\forall$ is unrestricted in $\mathcal{L}$ but restricted in $\mathcal{L}^+$ [‣W17: 102]. By classical lights, if $\forall$ ranges over exactly what $\exists^+$ does, inconsistency and so triviality follow; but that does not impugn the absolutist credentials of the view that a candidate absolutely general quantifier $\forall$ also is unrestricted in $\mathcal{L}^+$.

‣Q4 Absolutely generally quantified sentences are those whose quantifier is unrestricted from the perspective of any other (expanded) language. They are absolutely general because of their unrestrictedness no matter how the language is expanded.

7.3.3 Absolutism in a universal language. If this appeal to languages is considered undesirable, we could also make sense of considerations as to indefinite extensibility within a single language with respect to multiple quantifiers. The absolute generality of the absolutist's proposed quantifier, then, rests partly on the expressiveness of the language of whose signature the quantifier is part. For example, the universal quantifier in the theory of the naturals is not absolutely general, even though it conforms to the standard natural deduction rules. A provisional proposal is that the absolutist must identify a candidate quantifier in a sufficiently expressive language; but 'sufficiently' remains obscure.

The challenge to the absolutist is to either explain away the result of diagonalisation as nothing at all, or to explain how they could have quantified over it in the first place. What is essential to the absolutist is that there should be some way of referring to the putative result of diagonalisation (whether or not it exists). In the signature of set theory, we can write $\{x : x \notin x\}$, or $\delta(\Omega)$, or similar. If we limit ourselves to sets, then, what is important is that the signature should be rich enough to express and name the (putative) objects in question, and to express the putative rules governing them (for example, naïve comprehension). But this must apply not just to sets but to all concepts. We could therefore abandon any appeal to multiple languages; what is important is that the unrestricted quantifier is in a *suitably expressive* language. The demand that arises is that there should be some language in which all true theories can be stated, and that this language can accommodate an *unrestricted* quantifier—a universal language; its absoluteness arises from the expressibility of all true theories in the language.

‣Q4 Absolutely generally quantified sentences are those whose quantifier is unrestricted in a universal language. Their meaning is given by their unrestrictedness in a *universal* language.

The obvious argument against the availability of such a language is a language that is semantically closed ('expresses its own semantic concepts' [‣P84: 118]) must be inconsistent; to the dialetheist, this is welcome.

Are these definitions circular? Perhaps. But they are less obviously circular than the initial statement of absolute generality that motivated the demands I have attempted to meet above: mere insistence, without much elaboration, that the putatively absolutely general candidate quantifier indeed is absolutely general.

8 The inescapability of contradiction.

8.1 *Paradox lost?* The proposal above is not automatically dialetheist. What might motivate dialetheism if absolutism is articulated so? It is not susceptible to Russell's paradox. It does not rely on taking Cantor's theorem to be a dialetheia.

8.2 *Paradox regained.* However, the Williamson–Russell *reductio* does arise. It is still desirable to characterise interpretations, not least to characterise logical consequence, as explained in ‣¶ 6.3.1. Therefore, there is at least one candidate dialetheia, and so reason to accept paraconsistent logic. Indeed, there is reason to think that we have identified the 'right' dialetheia to accept, in that it is the hardest to avoid. I shall now expound Williamson's demand for generalisation over interpretations of a language for a second time, more carefully. What must be stressed is the parsimony of the resources on which Williamson calls in deriving a contradiction.

The Williamson–Russell *reductio* shows that *interpretations* are indefinitely extensible. What interpretations are there? What do they do? Predicate letters $P, Q, \ldots$ are intended to be quite general, and there is no obvious reason to restrict which 'contentful predicates' may be substituted for them—interpretations can then apply predicate letters to objects in accordance with the predicates so substituted.

Let us define an interpretation $I(F)$ that interprets the predicate letter P according to the following:

(18) For everything o, $I(F)$ is an interpretation under which P applies to o iff o Fs.

So far, all we have done is extended the notion of interpretation above without ontological commitment to sets or pluralities; we might as well have written homophonically:

(19) For everything o, Po is $\text{true}_{I(F)}$ iff o Fs.

Let us then define a verb R.

(20) For everything o, o Rs if and only if o is not an interpretation under which P applies to o.

If the notion of 'application' is regarded as obscure, we can instead write

(21) For everything o, o Rs if and only if o is not an interpretation under which Po is true_o.

Williamson writes that 'the naive theorist is committed to treating "R" as a contentful predicate, since it is well-formed out of materials entirely drawn from the naive theory itself' [‣w03: 426]. To be more explicit, these are absolutely general quantification, the connective iff, and the notion of application (or truth) under an interpretation. The only difference from the truth clauses borrowed from Studd above is that we also interpret predicates in the obvious way.

Accordingly, $I(R)$ behaves thus.

(22) For everything o, $I(R)$ is an interpretation under which P applies to o if and only if o is not an interpretation under which P applies to o.

Alternatively,

(23) For everything o, $I(R)$ is an interpretation such that Po is $\text{true}_{I(R)}$ if and only if o is not an interpretation under which Po is true_o.

Since 'everything' is absolutely unrestricted, 'o can be $I(R)$ itself', whence

(24) $I(R)$ is an interpretation under which P applies to $I(R)$ if and only if $I(R)$ is not an interpretation under which P applies to $I(R)$.

And, again alternatively,

(25) $I(R)$ is an interpretation such that Po is $\text{true}_{I(R)}$ if and only if $I(R)$ is not an interpretation under which Po is $\text{true}_{I(R)}$.

The relativist answer is to restrict the quantifier from quantifying over $I(R)$; in that case, ▸(24) and ▸(25) do not arise. The dialetheist answer is that ▸(24) and ▸(25) should be accepted but that that does not lead to triviality.

8.3 *The place of meaning.* Some objections to absolutism derive contradictions from the *meaning* of putatively absolutely general quantification. In the previous chapter, I considered the prospects of a set-theoretic approach: the meaning of a universally quantified statement is some claim about every member of the corresponding set-domain. The relativist will then derive some paradox from the alleged meaning of some absolutely generally quantified sentence—for example, on the set-domain account, the requirement for a universal set. This relativist objection (as stated) is unconvincing to the many absolutists who do not account for absolutely general quantification set-theoretically; and the same may apply to other accounts of the meaning of quantified statements.

The resulting impasse can only be addressed by asking what, if anything, would motivate accounting for domains as sets in the first place. The absolutist could even deny that they need to supply an account of meaning in the first place. Perhaps they could state their view as one on which some quantification is unrestricted and that such quantification does not lead to triviality (cf. ▸¶ 1.1). If that were the case, there would be no reason to think that contradictions derived by appeal to the meaning of putatively absolutely general quantification are of any concern. Such insistence runs the risk of ignoring a legitimate demand for a theory of the meaning of quantifiers. Analogously, relativists who do not address this concern run the risk of simply asserting that some theory of meaning for quantifiers must be supplied without stipulating why or what such a theory should do.

However, relativists can avoid that quagmire through arguments that trade on a combination of plausible principles about reasoning about quantifiers combined with an assumption there is an absolutely general domain will inevitably lead to triviality. Such an argument will begin by stipulating requirements on a putatively absolutely general quantifier to which, in principle, the absolutist should also assent; it could continue by articulating proof rules—for example, of natural deduction—acceptable to absolutists; and it will then derive triviality. It is possible to regard the development of Russell's paradox in a somewhat similar light. In that case, the task of developing those plausible requirements was fairly widely agreed on in the form of naïve set theory.

The Williamson–Russell *reductio* takes exactly this form. It *uses* an absolutely general quantifier to derive paradox, rather than considering what absolutely generally quantified statements might *mean* and showing those meanings to be paradoxical. It uses the absolutely general quantifier in a way that is to be expected if we are to characterise logical consequence. Accordingly, the Williamson–Russell *reductio* cannot be dodged simply by rejecting accounts of meaning above. Given that it still leads to contradiction, the case for dialetheism is strong.

Indeed, absent an account of logical consequence, it is unclear whether talk of quantification would be meaningful in the first place. It would, for example, be unacceptable for a candidate connective standing for 'and' to fail to license the inference of conjuncts from a conjunction, or for a candidate connective for 'or' to fail to license a disjunction from a disjunct. Similarly, an absolutist whose proposed absolutely general quantifier simply does not license, for example, universal instantiation is simply changing the subject.

8.4 *Proof-theoretic paradox.* We might wonder whether dialetheism can be avoided with a stronger form of inferentialism. Why not do away with interpretations and characterise logical consequence syntactically? We might call this position *proof-theoretic absolutism*, in the vein of *proof-theoretic semantics*. Indeed, Francez, who has with Ben-Avi offered *inter alia* proof-theoretic accounts of generalised quantifiers, suggests that the 'issue... [of] the possibility of quantifying over "absolutely everything"' is 'associated with entities in models' and 'accompan[ies] [model-theoretic semantics]' [▸F+15]. If this is taken to be adequate, why follow a dialetheist option at all? Can we derive something similar to the Williamson-Russell *reductio* in this environment? Yes, at least classically, although somewhat painfully—or so I shall claim.

The Williamson-Russell *reductio* begins with the observation that '[s]ooner or later the naive theorist will want to generalize over all (legitimate) interpretations of various forms in the language' [▸W03: 425]. What is the equivalent in the case of proof theory? The answer is presumably a proof-theoretic *context*; contexts $\Gamma \vdash \phi$ i.e. prove conclusions ϕ.[5] A context can be thought of, intuitively, as corresponding to a hypothesis, or set of assumptions. Williamson's interpretations in particular interpret *satisfaction*: they interpret interpretations either to satisfy or not to satisfy sentences. Accordingly, we must consider whether contexts prove that certain sentences prove sentences, and so the notion of proof here should not be thought to formally capture proof in full generality; for example, 'due to Löb's theorem, the language of any consistently recursively axiomatisable arithmetical theory $\mathcal{T}$ containing Peano Arithmetic cannot contain a formula for which the standard Hilbert-Bernays conditions on provability (to be listed soon) and reflection for the full language of $\mathcal{T}$ hold' [▸P+23: 205]. But we can appeal, for example, to Priest's notion of *naïve proof*: 'informal deductive arguments from basic statements', that is, statements 'known to be true without or having to look for a proof' [▸P06: § 3.2]. Or we could appeal to an incomplete but sound set of proof rules for some underlying consequence relation.

What is provable in which contexts is of just as much concern to the inferentialist semantic theorist as what is satisfied in which interpretations. Proof-theoretic semantics hardly makes less pressing demands for an account of logical consequence. In fact, it makes those demands if anything more pressing, since logical consequence is then prior to meaning, rather than definable from it model-theoretically. Any account of logical consequence will at least have to appeal to some notion along the lines of what are formally called 'contexts' and correspond approximately to the informal notion of assumptions.

Williamson has us consider an interpretation $I(F)$ under which we substitute some contentful predicate for F to interpret the predicate letter P such that for everything o, $I(F)$ is an interpretation under which P applies to o iff o Fs. We can analogously consider whether there is a *context* $\Gamma(F)$ such that for everything o, $\Gamma(F)$ *proves that P applies to o* iff *o F*s. Let us therefore attempt

(26) $\Gamma(F) = \{\forall o F(o) \leftrightarrow Po\}$

5. We do not need to concern ourselves with whether they can also prove multiple conclusions à la Gentzen.

In order to avoid worries arising from Löb's theorem, let us introduce an operator $\vdash'$, and consider what sort of rules are required for a paradox to follow. If $\Gamma(F) \vdash' \forall o(F(o) \rightarrow Po)$, and $\vdash'$ is closed under *modus ponens*, then if o Fs, $\Gamma(F) \vdash' Po$. So one direction is fairly easy, provided that $\vdash'$ is closed under *modus ponens*.

What about the other direction? We want to show that if o does not F, $\Gamma(F) \nvdash' Po$. Here, there is a disanalogy with interpretations. It is easy to define an interpretation that does not satisfy some sentence by ensuring that it satisfies the sentence's negation. But a context may prove both a sentence and its negation even classically—if it is inconsistent and so trivial. All the definition of $\Gamma(F)$ appears to deductively deliver is that $\Gamma(F)$ *proves the following*: if o does not F, P does not apply to o. The straightforward argument does not seem to work.

However, if $\vdash'$ is no stronger than $\vdash$, the argument goes through: it is fairly obvious that, taking $\vdash'$ to be a binary predicate, we do not have that $F(o) \vdash (\Gamma \vdash' Po)$ (since we do not have that $F(o) \vdash \sigma Ro$ for binary relations R and constants or variables σ in general.)[6] So what we require is a binary relation $\vdash'$ such that

(27) $\forall x \forall y \forall z(x \vdash' y \wedge y \rightarrow z \rightarrow x \vdash' z)$.

(28) There are no ϕ and ψ such that $\phi \vdash' \psi$ but $\phi \nvdash \psi$.

If we define $\vdash'$ as the minimal relation satisfying ▸(27), since $\vdash$ is also closed under *modus ponens*, $\vdash'$ naturally satisfies ▸(28).

Given this definition of $\Gamma(F)$, we can begin, following Williamson, by defining a *bona fide* context.

(29) For everything o, $\Gamma(F)$ is an a context that proves that P applies to o iff o Fs.

Then we define a verb R.

6. It suffices to provide a model of the sole premiss and the conclusion, and we can provide one on which the extension of P comprises exactly everything o that does not F. The appeal to a model is inessential in that it would also be possible to construct a maximal consistent set witnessing the same, even though the construction via a model is more convenient. There is no obvious reason for the proof-theoretic absolutist to renounce use of models as witnesses of the consistency of innocuous-seeming finite sets of sentences—that hardly commits one to model-theoretic definitions of logical consequence or truth conditions. In demanding that $F(o)$ is a *contentful* predicate, we exclude such pathological definitions as $F(o) =_{\mathrm{df}} \neg Po$, which would equally stymie the definition of $I(F)$. It is, however, contentful to say that o self-applies a particular predicate.

(30) For everything o, o Rs if and only if o is not a context that proves that P applies to o.

Taking R as the argument to Γ,

(31) For everything o, $\Gamma(F)$ is a context that proves that P applies to o iff o is not a context that proves that P applies to o.

Then, substituting $\Gamma(F)$ for o by universal instantiation,

(32) $\Gamma(F)$ is a context that proves that P applies to $\Gamma(F)$ iff $\Gamma(F)$ is not a context that proves that P applies to $\Gamma(F)$.

And, again, this is inconsistent, and would lead to explosion in classical logic.

The *proof-theoretic* absolutist is therefore put in exactly the same position as any other absolutist in the wake of the Williamson-Russell paradox, and there is no obvious reason to think that the considerations that lead to my proposed form of dialetheist absolutism would importantly vary.

An alternative formulation can be given in terms of *coherence*: $\Gamma \vdash \phi$ just in case it is incoherent, for example, to accept each of Γ and reject ϕ [▸R09].[7] Then we should like a context $\Gamma(F)$ such that

(33) for everything o, it is incoherent to accept $\Gamma(F)$ and reject Po just in case $F(o)$.

Again, one direction is easy: if $F(o) \to Po$ and $F(o)$ it is surely incoherent to reject Po. On the other hand, if $\neg Fo$, it is surely *coherent* to accept that $\neg F(o) \to Po$ and reject Po. Then we can substitute $F(o) \leftrightarrow Po$ to the same effect.

8.5 *Indefinite extensibility.* The Williamson–Russell *reductio* can be taken to show that interpretations are *indefinitely extensible* [▸S19: § 1.4]. Defining indefinite extensibility, however, is notoriously tricky; and absolutists could press the point in order to reject common relativist from indefinite extensibility. That could also, however, prove a problem for my proposed form of dialetheist absolutism: if my argument is of the same form as problematic relativist objections from indefinite extensibility, my motive for dialetheism will be correspondingly weakened.

The worry above is best dispelled by close examination of precisely how an absolutist could claim that objections from indefinite extensibility fail. I shall claim that both *reductiones* expounded

7. Thanks to Chris Scambler for pointing this out.

above—concerning interpretations and proof-theoretic contexts respectively—largely emerge unscathed from these objections.

A first absolutist response to indefinite extensibility would be to charge that it incorrectly supposes an 'all-in-one' principle [‣c94: 7].

> The general principle appears to be that to quantify over certain objects is to presuppose that those objects constitute a "collection," or a "completed collection"-some one thing of which those objects are the members. I call this the All-in-One Principle.
>
> ...It would be more accurate to speak of a battery of principles, varying in strength. According to one, the values of the variables of a first-order language must constitute a set; another requires only a class, perhaps ultimate; still another, designed to accommodate talk of all classes, requires only a hyper-class; and so on. But a common idea runs through them all: the values of the variables must be in, or belong to, some one thing.

Taken at face value, in saying that for any one or more things an interpretation or context applies *P* to exactly those objects, one does not commit oneself to an All-in-One principle: these objects no more *belong* to the interpretation than any other objects. But we might worry that the reasons to reject the All-in-One principle generalise to *all* universal singularisations, as Florio and Linnebo would put it.

The dialectic position is perhaps best understood from the opposite perspective: what exactly is required for the 'needs of quantification to be served'? Cartwright writes that 'those needs are already served by there being simply the cookies in the jar, the natural numbers, the pure sets; no additional objects are required'. This puts him in the same position as the pluralist absolutist; and, in common with everybody else, the pluralist absolutist really ought to be able to explicate logical consequence. We are then in familiar territory: by plural Cantor, a type-hierarchy beckons, and this leads to an expressibility deficit; to avoid that expressibility deficit, we require a sort of universal singularisation along the lines of an interpretation or context to characterise logical consequence, and the paradox beckons again. The All-in-One principle should not be assumed; but a variant can be justified.

It is also open to the absolutist to avail themselves of typed resources; this puts them in exactly the same position as discussed in ‣¶ 6.3—expressibility deficits are unavoidable.

9 Set theory and recapture reconsidered.

9.1 *The position resulting* from chapter ▸III and ▸§ 8 is that we can avoid many ontological commitments (e.g. to universal sets or pluralities) through sentential articulations of absolutism, but that the characterisation of logical consequence delivers contradiction once again. Accordingly, we should adopt a paraconsistent logic. This faces exactly the same difficulties outlined in ▸¶ 3.4, namely that much *quasi-valid* reasoning seems perfectly unobjectionable. I shall focus now on quasi-valid reasoning in set theory and mathematics. This is for four reasons reasons. First, it seems to be perhaps the most prominent difficulty for paraconsistent logicians. Second, it is arguably of sufficient difficulty that success here would be a good sign for success in recapturing other quasi-valid reasoning. And, third, orthodox mathematics is of sufficient generality and utility that it may be possible to use recaptured orthodox mathematics to recapture other reasoning. This of course leaves the argument in this section somewhat provisional vis à vis quasi-valid reasoning generally.

The fourth reason is that it might be thought that paraconsistent strategies in response to the paradoxes of absolute generality are particularly closely bound up with the set-theoretic antinomies of set theory, and so must radically revise both set theory and, thence, mathematics. Arguably this puts it in a particularly difficult position relative to paraconsistent responses to paradoxes of truth. What I sought to argue in ▸§ 6, and, more successfully, I hope, in chapter III, is that they are not in fact as closely connected as they seem. The strategy below may seem to be a fairly obvious attitude to set theory for a dialetheist motivated by, for example, the paradoxes of naïve truth. I hope to have shown that there is no reason that dialetheists equally concerned by absolute should avoid it.

I shall consider two approaches. On the first option, which I advocate, there is no quasi-valid reasoning in set theory and mathematics to recapture at all. Orthodox mathematics can be captured as perfectly valid *simpliciter* by paraconsistent lights. Therefore, no special manœuvres to accommodate quasi-valid reasoning are needed. On the second, quasi-valid reasoning is distinguished from invalid reasoning in general through pragmatic principles. I argue that this is less attractive.

9.2 *Non-naïve non-classical set theory.* The difficulty with paraconsistent naïve set theory is not, in fact, its paraconsistency. It is its naïveté.

Khomskii and Oddsson give a mathematically fruitful and non-trivial[8] paraconsistent *and* paracomplete set theory in the logic BS4 [▸K⁺23]. They first introduce the PZFC axioms.

9.2.1 The logic BS4 [▸K⁺23: § 2]. BS4 has the same signature as ordinary first-order logic, but ~ is used for negation, and there is a constant connective ⊥.

Suppose that τ is a signature with constant and relation symbols. A *T/F-model* $\mathcal{M}$ comprises a domain M, an element $c^{\mathcal{M}}$ for each constant symbol c, positive and negative interpretations $(R^{\mathcal{M}})^+ \subseteq M^n$ and $(R^{\mathcal{M}})^- \subseteq M^n$ for every relation R, a binary relation $=^+$ exactly coinciding with true equality, and a symmetric binary relation $=^-$.

I $\mathcal{M} \vDash^T (t = s)[a, b] \iff a =^+ b.$

$\mathcal{M} \vDash^F (t = s)[a, b] \iff a =^- b.$

II $\mathcal{M} \vDash^T R(t_1, \dots, t_n)[a_1 \dots a_n] \iff R^+(a_1, \dots, a_n)$ holds.

$\mathcal{M} \vDash^F R(t_1, \dots, t_n)[a_1 \dots a_n] \iff R^-(a_1, \dots, a_n)$ holds.

III $\mathcal{M} \vDash^T {\sim}\varphi \iff \mathcal{M} \vDash^F \varphi.$

$\mathcal{M} \vDash^F {\sim}\varphi \iff \mathcal{M} \vDash^T \varphi.$

IV $\mathcal{M} \vDash^T \varphi \wedge \psi \iff \mathcal{M} \vDash^T \varphi$ and $\mathcal{M} \vDash^T \psi.$

$\mathcal{M} \vDash^F \varphi \wedge \psi \iff \mathcal{M} \vDash^F \varphi$ or $\mathcal{M} \vDash^F \psi.$

V $\mathcal{M} \vDash^T \varphi \vee \psi \iff \mathcal{M} \vDash^T \varphi$ or $\mathcal{M} \vDash^T \psi.$

$\mathcal{M} \vDash^F \varphi \vee \psi \iff \mathcal{M} \vDash^F \varphi$ and $\mathcal{M} \vDash^F \psi.$

VI $\mathcal{M} \vDash^T \varphi \to \psi \iff$ if $\mathcal{M} \vDash^T \varphi$ then $\mathcal{M} \vDash^T \psi.$

$\mathcal{M} \vDash^F \varphi \to \psi \iff \mathcal{M} \vDash^T \varphi$ and $\mathcal{M} \vDash^F \psi.$

VII $\mathcal{M} \vDash^T \varphi \leftrightarrow \psi \iff (\mathcal{M} \vDash^T \varphi$ if and only if $\mathcal{M} \vDash^T \psi).$

$\mathcal{M} \vDash^F \varphi \leftrightarrow \psi \iff (\mathcal{M} \vDash^T \varphi$ and $\mathcal{M} \vDash^F \psi)$ or $(\mathcal{M} \vDash^F \varphi$ and $\mathcal{M} \vDash^T \psi).$

VIII $\mathcal{M} \vDash^T \exists x \varphi(x) \iff \mathcal{M} \vDash^T \varphi[a]$ for some $a \in M.$

$\mathcal{M} \vDash^F \exists x \varphi(x) \iff \mathcal{M} \vDash^F \varphi[a]$ for all $a \in M.$

IX $\mathcal{M} \vDash^T \forall x \varphi(x) \iff \mathcal{M} \vDash^T \varphi[a]$ for all $a \in M.$

$\mathcal{M} \vDash^F \forall x \varphi(x) \iff \mathcal{M} \vDash^F \varphi[a]$ for some $a \in M.$

X $\mathcal{M} \vDash^T \bot \iff$ never.

$\mathcal{M} \vDash^F \bot \iff$ always.

Then for any set of formulæ σ and formula ϕ, $\Sigma \vdash_{\text{BS4}} \phi$ just in case, for every T/F-model $\mathcal{M}$, if $\mathcal{M} \vDash^T \Sigma$ then $\mathcal{M} \vDash^T \phi$.

8. relative to ZFC.

The truth value of a formula is given by

$$[\![\phi]\!]^{\mathcal{M}} = \begin{cases} 1 & \mathcal{M} \models^T \phi, \mathcal{M} \not\models^F \phi \\ \mathfrak{b} & \mathcal{M} \models^T \phi, \mathcal{M} \models^F \phi \\ \mathfrak{n} & \mathcal{M} \not\models^T \phi, \mathcal{M} \not\models^F \phi \\ 0 & \mathcal{M} \not\models^T \phi, \mathcal{M} \models^F \phi \end{cases}$$

We then define so-called *strong (bi-)implication*: $\phi \Rightarrow \psi$ abbreviates $(\phi \to \psi) \wedge (\psi \to \phi)$, and $\phi \Leftrightarrow \psi$ abbreviates $\phi \leftrightarrow \psi \wedge ({\sim}\psi \leftrightarrow {\sim}\psi)$. We also define *classical negation*: $\neg\phi$ abbreviates $\phi \to \bot$; $!\phi$ abbreviates ${\sim}\neg\phi$ ('presence of truth'—dependent only on whether ϕ was true) and $?\phi$ abbreviates $\neg{\sim}\phi$ ('absence of falsity'—dependent only on whether ϕ was false).

9.2.2 *Non-naïve non-classical sets* [▸K+23: § 3]. We can think of paraconsistent and paracomplete sets as sets x for which we no longer require that the positive extension (sets y such that $y \in x$) and negative extension ($y \notin x$) are mutually exclusive (paraconsistent) and jointly exclusive (paracomplete). The positive extension is a set, but there is an asymmetry, since the negative extension is a proper class; however, the complement of the negative extension is a set, denoted the ?–extension. We can define $x^! =_{df} \{y :!(y \in x)\}$ and $x^? = \{y :?(y \in x)\}$. We shall abbreviate $!\phi \leftrightarrow ?\phi$ as $\circ\phi$.

We can abbreviate $\forall y(y \in x \Leftrightarrow \phi(y))$ as $x = \{y : \phi(y)\}$. Note that then the Russell set 'cannot be a set by the usual argument'.

9.2.3 PZFC [▸K+23: § 4]. The axiomatisation of PZFC—a variant of ZFC in BS4—is not wholly straightforward; I shall give only a few excerpts.

I *Extensionality* is given as $\forall x \forall y(x = y \Leftrightarrow \forall z(z \in x \Leftrightarrow z \in y))$. The first strong bi-implication is justified on the basis that 'extension seeks to define…meaning' and so 'needs to talk about both *truth* and *falsity*'; in a system in which truth and falsity are not interdefinable, this is a plausible requirement. The second strong bi-implication is justified on the basis that it is necessary to equate both the !– and ?–extensions.

II *Comprehension* is given as $\forall u \exists x \forall y(y \in x \Leftrightarrow y \in u \wedge \phi(y))$. By strong implication, we have, for example, that $x = \{y \in u : \phi(y)\}$ is a set.

III *Classical supersets.* We can write that a set is 'classical'—*i.e.* its !– and ?–extensions are the same—with $\forall y \circ (y \in C)$. Then the classical superset axiom is that $\forall x \exists C(x \subseteq C \wedge \forall y \circ (y \in C))$, where $\phi \subseteq \psi$ abbreviates $\forall x(x \in \phi \Rightarrow x \in \psi)$.

From this it can be derived that the !– and ?–extensions of sets are themselves sets.

9.2.4 *Several connexions to* ZFC can then be proved.

5.1 *Theorem.* PZFC and $\forall x(x^! = x^?)$ is equivalent to ZFC.

Proof. 'If every set is classical, then there is no distinction between ~ and ¬, nor between → and ⇒. Likewise, ! and ? can be discarded. The Classical Superset Axiom is trivial. SO what remains of PZFC is precisely the collection of ZFC axioms.' □

5.2 *Theorem.* Suppose that there is an inconsistent and an incomplete set. Then for all classical sets u and v there is a set such that $x^! = u$ and $x^? = v$.

5.3 *Definition.* BZFC is the combination of PZFC and the *anti-classicality axiom* $\exists x(x^! \nsubseteq x^?) \wedge \exists x(x^? \nsubseteq x^!)$.

5.4 *Remark.* We can axiomatise a paraconsistent but complete set theory with PZFC and $\exists x(x^! \nsubseteq x^?) + \forall x(x^? \subseteq x^!)$.

§ 6 It is possible to define tuples, Cartesian products, and relations, with slight subtleties.

7.5 *Corollary.* If ZFC is consistent, BZFC is non-trivial.

9.1 *Theorem.* BZFC—the result of adding an anti-classicality axiom to PZFC—and ZFC are bi-interpretable.

9.3 *Axiomatic recapture.* The account above does not advert to any notion of quasi-validity or defeasible pragmatically acceptable inference. It uses a perfectly paraconsistent logic and delivers a theory bi-interpretable with ZFC deductively. It does not then need to characterise a separate class of *quasi-valid* reasoning and attempt to explain why in some cases a contradiction turns out to be a dialetheia and in other cases it turns out to be the basis of a *reductio*. On the other hand, it significantly revises the axioms of set theory. Let us call this form of revision and the associated project *axiomatic recapture*, in contrast to *logical recapture*. It is natural to ask whether axiomatic recapture suffers from similar difficulties in motivating the axiomatic revisions necessary.

There are several ways of justifying the use of BZFC. One answer is a 'really full-blooded Platonism' [‣B99]: '[e]very mathematical theory—consistent and *inconsistent* alike—truly describes some part of the mathematical realm [provided that it is nontrivial]'. So BZFC describes a non-trivial part of 'platonic heaven' [▸B99] just as much as ZFC does, and is in similarly good standing. It is part of the 'set-theoretic multiverse', along with ZFC, and ZFC+CH, and so on [▸H12].

But ZFC is not merely a part of platonic heaven in good standing. It plays a more systematic rôle in mathematical theorising. Theorems derivable from ZFC are taken by the mathematical community to be theorems *simpliciter*. Theorems derivable from ZFC and the continuüm hypothesis, for example, are not; or they are taken to be conditional: if CH is true, then some conclusion follows. ZFC can even override intuition: Banach-Tarski is in better standing than naïve comprehension, even though the latter is considerably more pre-theoretically intuitive than the former.

If ZFC does not deserve this special rôle, there is no need to argue that BZFC deserves it either; all that is required is to show that they legitimately occupy their places in Platonic heaven. The trickier task is to characterise the place of BZFC if ZFC does deserve this special rôle. The question is then perhaps this: with a background logic of BS4, can BZFC attain a status similar to that of ZFC with a background of classical logic? That question is to be posed on the assumption that ZFC genuinely does deserve its special status. If so, the paraconsistent logician is in a position to ground mathematics in BZFC precisely as the classical logician does with ZFC. This ensures that all orthodox mathematics is just as respectable[9] by paraconsistent lights as by classical lights; and, in turn, it rescues all applications of mathematics in the natural sciences. We mustn't just ensure the admission of BZFC to platonic heaven—which, I think, should be fairly uncontroversial to adherents of ZFC; wherever, by classical lights, ZFC sits, BZFC must sit by nonclassical lights.

If the dialetheist can justify such a view, it seems that a central argument against paraconsistent logic—for example, Studd's sole argument—can successfully be refuted. There is no *mathematical* loss to be incurred. (I assume here that ZFC is consistent, and therefore that BZFC is nontrivial.) If the classical logician can independently motivate a view that sets must be classical, the dialetheist can simply accept the anti-classicality axiom, and so regain precisely ZFC.

It is common to distinguish between *intrinsic* and *extrinsic* justifications of set-theoretic axioms; so Maddy claims that some set theory texts purport to derive ZFC 'directly from the concept of set', so that its axioms are 'somehow "intrinsic" to it (obvious, self-

9. Perhaps excluding category theory.

evident), while other axiom candidates are only supported by weaker, "extrinsic" (pragmatic, heuristic) justifications, stated in terms of their consequences, or intertheoretic connections, or explanatory power, for example' [▸M88: 482–3].

To the extent that ZFC is intuitive, most of the considerations behind BS4's reaxiomatisation make sense: for example, there is intuitive reason to equate both the !– and ?–extension in the axiom of extensionality when the two vary independently. So intrinsically ZFC and BZFC seem to occupy the same position under classical and non-classical logic respectively.

Maddy convincingly argues that this is at least historically inaccurate: 'the first axioms for set theory were motivated by a pragmatic desire to prove a particular theorem' [▸M88: 483]. Moreover, theoretically, intuitive force *per se* is hardly a good guide: classical naïve set theory surely does better than ZFC, and yet is trivial. What, then, about extrinsic justifications? Maddy goes so far as to argue that these should have priority over intrinsic justifications [▸M11: V.4]. In considering why intrinsic justification is valuable, we might think that 'it's extremely useful to have a workable heuristic picture of the sort of thing we're investigating mathematically', and, where one is absent from a theory, 'we're reluctant to even pursue it'—for example, in the case of New Foundations; if the result is mathematically fruitful, it's 'rational…to try to extend it in ways that seem "natural" or harmonious with its leading intuitions'. Both, however, are grounded in the prospect of further mathematical progress—an extrinsic motivation.

If we evaluate a logico-mathematical package as a whole, by deductive consequence, PZFC with anti-classicality and BS4 is indistinguishable from ZFC with classical logic; and so there is no reason to prefer one over the other.

It might be objected that a logico-mathematical package should not be evaluated as a whole: its individual constituents (logical and mathematical) should be evaluated in isolation. Perhaps paraconsistent logic considered *per se* is too weak. But it would be odd to demand that logic *per se* should deliver set theory. It might be thought that a neologicist account of *sets* can be provided, but even the neologicist *par excellence*, Bob Hale, is doubtful that that applies to ZFC: 'Zermelo–Fraenkel set theory…is not plausibly viewed as a purely logical theory, owing to the very substantial existence assumptions it involves' [▸H01: § 2], and, in any case, the neologicist faces the not insubstantial bad company problem amongst others.[10] It is hard to see how even classical logic could meet such a demand.

It seems that the dialetheist can pay a very modest price for paraconsistency; mathematical and set-theoretic orthodoxy need not be given up. Perhaps the central difficulty is deciding how and when to apply classical mathematics; here, the spectre of adhocness returns. Semantics in the vein of empirical linguistics, for example, could well be carried out in ZFC; but the semantics of quantifiers would then be restricted to the fragment of the language that does not include absolutely general quantifiers. Nevertheless, the classical logician faces a similar problem in justifying their use of ZFC and classical logic; and I suspect that any satisfactory answer they could give is one that the dialetheist could claim. Otherwise, the dialetheist's approach is only as *ad hoc* as the orthodox mathematician's; the only difference is that the *adhocness* of their approach is more visible because the alternative is better elucidated and less obviously disastrous in BS4 than in classical logic.

9.4 *Priest's pragmatic recapture compared.*

9.4.1 Priest gives the following general

> **Methodological Maxim (M)**[:]
> [u]nless we have specific grounds for believing that the crucial contradictions in a piece of quasi-valid reasoning are dialetheias, we may accept the reasoning.

We can take this to respond to the charge of adhocness. Of course, when we have (sufficiently strong) grounds for believing that the crucial contradictions are dialetheia, we should not accept the reasoning, since quasi-valid reasoning from dialetheia leads to triviality.

9.4.2 For reasons I shall put beyond the scope of this thesis, we might also accept

> **Principle R**[:]
> [i]f a disjunction is rationally acceptable and one of the disjuncts is rationally rejectable, then the other is rationally acceptable.

9.4.3 *Theorem* [‣P95: n 7, p. 130; ‣P06: Theorem 0, § 8.6]. Let Σ be a set of sentences and α a formula. Then $\Sigma \vdash \alpha$ (classically proves) iff, for some β, $\Sigma \Vdash \alpha \vee \beta!$ (paraconsistently proves).

10. See [‣S16] for a 'dynamic' way out.

The disjunctive syllogism is only one instance of quasi-valid reasoning. But the theorem above makes principle R applicable to all classical reasoning.

9.4.4 Suppose that $\Sigma \vdash \alpha$. By the theorem above, $\Sigma \Vdash \alpha \vee \beta$! By principle M, we defeasibly assume that β is not a dialetheia. By principle R, we then rationally accept the other disjunct, α.[11]

Axiomatic recapture is preferable for at least two reasons. First, we no longer need to ask whether there are 'specific grounds for believing that the crucial contradictions in a piece of quasi-valid reasoning' are dialetheia. Second, principle R, whether or not it is well-motivated, is ultimately pragmatic; axiomatic recapture puts its results on a firmer footing, perhaps more appropriate to the epistemic status of mathematics.

However, the tenability of axiomatic recapture of ZFC rests on a system in which consistency is expressible. Priest, mistakenly, assumes that '[t]here no statement that can be made which *forces*' a sentence to 'behave consistently' [▸P06: 112]; if that were the case, Priest's proposal would be preferable, since it makes no appeal to such a notion. Consistency operators can be introduced without triviality [▸O15], and one was employed above in the characterisation of BZFC. But we might still consider them problematic [▸R21], in which case Priest's approach would retain its advantages. I shall now argue that Rosenblatt's argument does not apply to axiomatic recapture, and so the use of a consistency operator is acceptable.

The central problem Rosenblatt identifies is that 'if both truth and consistency are expressible, then there will be a statement p such that $p =_{\mathrm{df}} \circ p \wedge \neg Tp$' [▸R21: 35]. Rosenblatt initially identifies three alternatives: first to 'eschew the unrestricted Tarski schema'; second, to take consistency to be 'a meta-linguistic notion, not expressible in the object language'; and, third, to argue 'that consistency is indeed expressible in the object language, but not consistently'. A fourth option, following Priest, is to 'separate in the object language the statements that are consistent from the statements that are not', and to accept 'the instance of explosion for ϕ... if and only if ϕ is consistent' [▸R21: 36]. Rosenblatt rejects this view on the basis that there 'are many maximal non-trivial sets of instances of the rule of explosion that the paraconsistent theorist could in principle endorse. If she is solely guided by the demand of non-triviality, then she will have no principled way of

11. Priest does not seem to explicitly articulate this line of reasoning, but it seems to be the obvious way of combining the steps above.

choosing one of these non-trivial sets'. What should we make of this argument in light of the previous discussion?

I agree with Rosenblatt that we may not have principled grounds to choose any particular maximal non-trivial set. But the scope of the recapture above was not justified on the basis of maximality. It requires two premisses: first, that *some* recapture is *formally* possible (with which Rosenblatt takes no issue)—in particular, enough to recapture ZFC; and, second, that ZFC is special enough to recapture. There may be underdetermination beyond that point, but underdetermination beyond a well-motivated core is not a good reason to reject that core of recapture. Indeed, one option is to accept a *minimal* 'non-trivial set of instances of the rule of explosion', corresponding to that core: it may possible to formally achieve such a result by restricting the classically motivated connectives ($\neg, \Rightarrow, \circ, \ldots$) to apply to set-theoretic statements only.

There are therefore two possibilities, depending on whether or not the special status of ZFC is justified. If its special status is justified, BZFC is also special, and so also may be specially entitled to make use of a consistency operator. If it does not have special status, there may be nothing wrong with describing one part of Platonic heaven that behaves as described by axioms containing a consistency operator; we need not think that this has any broader significance. Perhaps more importantly, if ZFC is not entitled to special status, it is unclear why failure to recapture it should be held against the dialetheist in the first place. Dialetheist absolutism, therefore, is not committed to problematic *mathematical* revision.

10 Conclusion.

10.1 The principal lines of argument.

10.1.1 The viability of naïve set-theoretic dialetheist absolutism remains to be shown, given current developments in paraconsistent naïve set theory (▸§ 5). The most viable form would be based on Weber's paraconsistent naïve set theory, but even then, the grounds for accepting the underlying logic are unclear.

10.1.2 At least two forms of dialetheist absolutism are tenable: plural dialetheist absolutism (▸§ 6), and sentential dialetheist absolutism (chapter ▸III).

10.1.3 A robustly reconciliatory attitude to orthodox set theory and mathematics is available to dialetheists, even when motivated by para-

doxes of absolute generality. Paraconsistent and paracomplete ZFC (▸§ 9) at least deductively delivers all theorems proponents of set-theoretic and mathematical orthodoxy could demand. If those demands are well-motivated, axiomatic recapture is at least as justified as classical axiomatisations of set theory are.

10.1.4 Plural and sentential dialetheist absolutism, on my preferred formulation, avoid the expressibility deficits incurred by 'traditional absolutism' whether typed or plural (▸¶ 6.3).

10.1.5 Sentential dialetheist absolutism may avoid a charge of adhocness that plural dialetheism faces, viz. to explain why set formation is not an acceptable universal singularisation but (for example) interpretations are, and as to why only some pluralities are coextensive with sets. (▸§ 6). Sentential dialetheist absolutism does not immediately have to answer such questions, since it makes no appeal to pluralities in the first place, and it has no need of a systematic theory of plurals.

10.2 *Remaining tasks for the dialetheist absolutist.* I leave these beyond the scope of the thesis, but any definitive assessment of the merits of dialetheist absolutism would have to return to them.

10.2.1 Other objections. The wider debate in which this thesis is situated is between absolutists and relativists (▸Q1). If objections from indefinite extensibility are the primary motive of relativism (▸Q2), whether or not they succeed is decisive (▸Q3). But the other objections that I preliminarily discounted in ▸§ 2 would need to be more fully dismissed.

10.2.2 The exact price of relativism needs to be assessed and compared. Recall that first attempt to state generality relativism in ▸¶ 1.2 succumbed to paradox; dialetheism may open the way not just for certain forms of absolutism but certain forms of relativism.

10.2.3 Classical set-theoretic revisionism. It is possible to ensure the availability of a universal set by classical axiomatic revisionism. Options include Quine's New Foundations, which 'has been profitably used to account for phenomena which seemingly involve large collections, most notably to give a foundation for category theory in set theory' [▸I22], and Button's Boolean Level Theory [▸B22].

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